THE INTERNATIONAL CONGRESS OF MATHEMATICIANS IN PARIS.

At the Zürich Congress of 1897 it was agreed to hold the next congress in Paris in 1900, the French Mathematical Society being charged with the preparations. Circulars have been issued at intervals during the last eighteen months, calling the attention of mathematicians to the arrangements in progress. The congress was finally announced for August 6th–11th, and the opening general meeting was held in the Palais des Congrès, in the Exhibition grounds, at 9.30 on the morning of Monday, August 6th. M. Poincaré was elected President, M. Hermite, who of course was not present, being the *Président d'honneur*. The executive board was constituted as follows: vice-presidents, MM. Czuber (Vienna), Geiser (Zürich), Gordan (Erlangen), Greenhill (London), Lindelöf (Helsingfors), Lindemann (Munich), Mittag-Leffler (Stockholm), Moore (Chicago, absent), Tikhomandritzky (Kharkoff), Volterra (Turin), Zeuthen (Copenhagen); secretaries, MM. Bendixson (Stockholm), Capelli (Naples), Minkowski (Zürich), Ptaszycki (St. Petersburg), Whitehead (Cambridge, absent); general secretary, M. Duporcq (Paris). After the announcement of the officers of the sections and the names of the official delegates, and a very few words from the President, the two addresses of the day, both in French, were delivered by MM. M. Cantor (Heidelberg) and Volterra (Turin); each occupied about three-quarters of an hour.

M. Cantor: *Sur l'historiographie des mathématiques.*

During the century drawing to its close the character of mathematics has changed; its devotees are now differentiated into geometers, analysts, algebraists, arithmeticians, astronomers, theoretical physicists, and historiographers. These last make no claim to advancing the science itself; they press neither towards the arctic pole of the theory of functions, nor towards the antarctic pole of algebra; they explore neither the steep surfaces of geometry nor the depths of differential equations. Their task is rather to draw up guides and charts, to indicate by what routes the results have been obtained, and what important points have been passed by without sufficient exploration. This work began with the History of Eudemus of Rhodes, B. C.

300, of which only a fragment has been preserved, just sufficient to excite lively regret for the loss of the whole. During the next two thousand years there were many bald chronicles of mathematics, but historiography as a science begins with Montucla. Notwithstanding the errors, unavoidable at that time, to be found in the two volumes of his Histoire des mathématiques (1st edition, 1758, 2d edition with two volumes by Lalande, 1799), Montucla "est encore et restera peut-être toujours un modèle que tout historiographe des sciences doit suivre." Kästner published four volumes of his Geschichte der Mathematik in the last four years of his life, 1796–1800. He has been alternately over-praised and depreciated; Gauss referred to him as the best poet among the mathematicians, the best mathematician among the poets of his day. His history is no real history, it is rather a catalogue *raisonné*, but it is nevertheless invaluable, on account of its conscientious analysis of a number of works, which, with their authors, would be otherwise absolutely unknown to us now. At about the same date, 1797–1799, there appeared the two volumes of Cossali's Storia critica dell'algebra, dealing exhaustively with the period 1200–1600; as regards Italy only, it is true, but then during this period the Italian algebra was of importance far surpassing that of any other country. Cossali's labors for the elucidation of Leonard of Pisa and Cardan are of special merit.

Bossut published in 1810 his Histoire des mathématiques; in this he gives only rapid *aperçus* of the general development, interesting to those that know already, useless to those that need to learn. In the present century we have first Chasles, to whom the speaker paid a warm personal tribute. In his Aperçu historique published in 1837, the notes, dealing with geometry, calculation, algebra, mechanics, which attain the dimensions of memoirs, form the model part of the volume, the text, the actual "Aperçu," being but a very condensed statement of the history of synthetic geometry. The other historical work of Chasles, the Rapport sur les progrès de la géométrie of 1870, is seriously affected by his ignorance of the German language. The years 1837–1841 saw the publication of Libri's Histoire des sciences mathématiques en Italie, from the earliest times up to the middle of the 17th century, a work which owing to the author's admirable style "se lit comme un roman, même dans les parties où elle n'en est pas un." Notwithstanding Libri's immense services in the study of manuscripts, his history is vitiated, as a historical

work, by his misplaced patriotism; according to him all progress in mathematics is due to the Italians, with perhaps a few scattered French writers. When he finds an Italian in possession of any ideas or methods, no matter whence derived, he at once credits him with their discovery. In any case, it is not possible to give any true idea of the history of mathematics by tracing it in one country only. If there is an international science, it is mathematics; it bears no stamp of nationality. In considering the earliest times, it is impossible to understand the course of mathematics in one country without following it in others also; to understand Greek mathematics, we must know something of Egypt and Babylonia; the mathematics of the Arabs cannot be explained without reference to Egypt, Greece, and India. After the invention of printing, so long as Latin was in use, mathematics had no country; and even when the frontiers were faintly marked by the use of different languages, they were speedily obliterated for most mathematicians.

Passing rapidly over Gerhardt and Quetelet, with a few words of recognition, M. Cantor spoke of Nesselmann's Die Algebra der Griechen, 1842, "un chef d'œuvre digne d'être mis a côté de l'Aperçu historique de Chasles"; of Arneth's Geschichte der reinen Mathematik, 1852, which would have been an excellent book, if the author had made a better apportionment of his space to his material—parts of the work "fourmillent de remarques aussi spirituelles que profondes"; of Hankel's posthumous fragment, 1876, "un torse d'une telle beauté qu'il eut été pitié de ne pas le mettre au grand jour"; and of the prince Baldassare Boncompagni's disinterested labors on behalf of historiography. In this sketch he passed over many authors "tous aussi morts que leurs livres; gardons-nous de les ressusciter"; and avoided all mention of living authors for very obvious reasons. He brought his address to a close by a forecast of the mode in which the history of more recent mathematics must be written. Regarding Lagrange as the founder of modern mathematics, this gives 1759 as the starting point; and from this year on, the different subjects will have to be treated in special volumes. This however will be insufficient; the development of the lines of thought that run through all these different branches of mathematics must be traced in one final volume, the History of Ideas; difficult to write, certainly, but indispensable, for as Jacobi said, "Mathematics is a science of which it is impossible to understand any one part without knowing all the others."

V. VOLTERRA: *Trois analystes italiens; Betti, Brioschi, Casorati.*

The scientific existence of Italy as a nation dates from a journey which Betti, Brioschi, and Casorati took together in the autumn of 1858, with the object of entering into relations with the foremost mathematicians of France and Germany. It is to the teaching, labors, and devotion of these three, to their influence in the organization of advanced studies, to the friendly scientific relations that they instituted between Italy and foreign countries, that the existence of a school of analysts in Italy is due.

The extent of their joint influence, affecting minds of many diverse casts, is largely due to the differences in their natural faculties, in the circumstances of their lives, and in their acquired tendencies. Brioschi, "toujours jeune par son caractère et toujours mûr par son esprit," a Lombard by birth, was at first an engineer; but at an early age he acquired a profound knowledge of the classical mathematical works, and was called to the chair of mechanics at Pavia at the age of 25. He founded the Polytechnic School at Milan, and held the directorship until his death; in his capacity of Senator, he was active in public affairs; he found time to engage in public works and in engineering; and up to the last, as Director of the *Annali di Matematica*, and President of the Accademia dei Lincei, he was one of the leaders of the mathematical movement in Italy. A great contrast to this active life is offered by the calm existence of Betti. He was born in a mountain village in Tuscany; at 34 he became a professor in the University of Pisa, and at 41 Director of the Scuola normale superiore of Pisa, whose organization is much like that of the École normale supérieure of Paris; he took no part in political movements. He loved scientific researches for their own sake exclusively, without regard to the results they might produce in the scientific world, or to their importance in teaching. He did not care for publishing his researches; and even when he did undertake this, he was apt to push it aside, attracted by new ideas. The knowledge that his intellectual conception could be realized was all-sufficient for him; he did not give himself the trouble of carrying it out in detail. When once he had obtained a clear vision of hidden truths, and had constructed in his own mind a system in which they proceeded directly from the simplest principles, "tout était fait pour Betti."

Casorati was born and lived at Pavia; he passed through the various grades in the University, where at the time of

his death he was professor of infinitesimal analysis. He lived and worked almost exclusively for his pupils; all his works bear the stamp of the practical teacher, bent on elucidating some obscurity, correcting some error, expounding some theory. All his writings were in a definite relation to his university teaching; in his mind there was no distinction between the work of the savant and the work of the professor.

The fundamental differences in the three can be brought out most clearly by a comparison of their attitude towards the theory of functions. The development of this theory exhibits three well-marked periods, corresponding to the three distinct phases that can be recognized in the history of any mathematical subject; these three phases, however, correspond also to three distinct modes of regarding questions in analysis, each of which has its advocates. In the first instance, the discovery of facts is all-important, and particular theories are elaborated. There are no uniform methods; every question is attacked on its own merits, and methods are created as occasion arises; the ideas and results disengage themselves finally from long calculations. In the theory of functions this manifests itself in the heroic period, personified in Euler, Jacobi, Abel; and this manner of approaching questions is natural to Brioschi, the engineer and practical man, with his extraordinary gift for dealing with formidable calculations. He remained faithful to the classical method, never attracted by the second phase, which he even scorned somewhat. In this second phase, ideas replace calculations; the philosophic spirit takes control and demands a general method including the whole subject in one body of doctrine. This desire found its fulfilment in the second period of the theory of functions, in the works of Cauchy, Weierstrass, and Riemann, who derive everything from the very sources of the fundamental conceptions. To this period belongs Betti the philosopher. His broad and cultivated mind loved philosophic systems; his Tuscan indolence (which is not intellectual idleness) caused him to delight in meditation rather than in mechanical labor. Curiously enough, his name is associated with the theory of Weierstrass just as surely as with that of Riemann; his education had made him an algebraist while nature meant him for a physicist.

In the final period the theories find their appropriate applications, their most suitable forms; they are refined by criticism, and cast into a didactic mould. The name of Casorati, critic and teacher, is associated with this third

phase. His work, Teorica delle funzioni di variabili complesse, has served more than any other one book to popularize in Italy the fundamental conceptions of the theory of functions, for the reason that, while reading it, difficulties disappear. The influence of this book is not confined to professed analysts; anyone attempting to trace the development of mathematics in Italy during this half century will find that analysts and pure geometers have influenced one another. For instance, the ideas of Riemann are at the foundation of many of the works of Italian geometers, and while the actual introduction of these ideas was due to Betti, it is this book of Casorati's that has carried them everywhere and attracted the attention of geometers.

This comparison of the work of these three analysts in the region that they had in common gives no idea, however, of the extent of the labors and influence of each one. For this it would be necessary to dwell on the work of Casorati in the theory of differential equations, in analytical and infinitesimal geometry; of Betti in mathematical physics and algebra, he being one of the first to accept the new ideas of Galois; of Brioschi in mechanics, algebra, and geometry. The field in which Betti and Brioschi first obtained renown was in fact that of algebra; their names will always be associated with that of Kronecker as second only to Hermite in their work on the equation of the fifth degree, an equation whose complete solution was due to and secured immortality for M. Hermite.

This concluded the business of the first general meeting, with the exception of one or two formal announcements relating to secretarial matters. This was the only one of the meetings to be held in the Exhibition grounds; all the others were held at the Sorbonne. Six sections were organized for the presentation of special papers, to meet on the 7th, 8th, 9th, and 10th of August, as follows:

Section I, Arithmetic and Algebra; Tuesday, Thursday and Friday mornings; president, M. Hilbert, secretary, M. Cartan.

Section II, Analysis; Tuesday and Thursday mornings; president, M. Painlevé, secretary, M. Hadamard.

Section III, Geometry; Tuesday and Thursday afternoons; president, M. Darboux, secretary, M. Niewenglowski.

Section IV, Mechanics and Mathematical Physics; Tuesday and Thursday afternoons; president, M. Larmor, secretary, M. Levi-Cività.

Section V, Bibliography and History; Wednesday morning and afternoon and Friday morning; president, Prince Roland Bonaparte, secretary, M. d'Ocagne.

Section VI, Teaching and Methods; Wednesday morning and afternoon and Friday morning; president, M. Cantor, secretary, M. Laisant.

Sections V and VI, however, amalgamated and sat as one section, thus making the sections the same as those at Zürich. It hardly seems advisable to give a complete list of the papers, as all will be given in the full official report, which will appear shortly; it seems better to give some account of the most interesting.

In Section I the most noticeable communication was that of M. Henri Padé, of Lille, "Aperçu sur les développements récents de la théorie des fractions continues." The object of this communication was the discussion of the question as to what is to be understood by the development of a function as a continued fraction, and the examination of the consequences of the answer obtained. For the function e^z, for instance, five such developments are already known, due to Euler, Lagrange, and Gauss. Why five? By what are the five characterized? How are they related to one another? The bond that unites them consists in a certain property of approximation common to their convergents, and thus there arises the more general question of the study of rational fractions satisfying this condition of approximation for a given function.* This investigation yields the following results. Every function that is developable by Maclaurin's formula gives rise to an infinity of developments as a continued fraction; as to form, these fractions present the common characteristics that all the partial numerators are monomials, with coefficient and exponent different from zero; as to matter, they are characterized by the property that all the convergents satisfy the condition of approximation mentioned above, and give an approximation whose order increases constantly as we pass from any convergent to the following one.

Among these fractions, called *fractions continues holoïdes* of the function, are included the regular continued fractions, and it is among these last that are found, as a very particular case, the developments in continued fractions already known for some special functions, for example, the five developments of e^z.

* "Sur la représentation approchée d'une fonction par des fractions rationnelles," *Ann. scient. de l' École norm. sup.*, vol. 9 (1892).

M. Padé then indicated the two ways in which these results can be generalized, the extension they involve in all the applications of continued fractions hitherto made, and the important consequences to which they lead, both in the theory of functions, where they have already introduced the question of the use of divergent power-series, and in the theory of numbers.

In Section II, the first paper read on Tuesday morning was M. Tikhomandritzky's "L'évanouissement des fonctions θ de plusieurs variables indépendantes." The function $\theta\left(u_k \underset{1}{\overset{p}{-}} \int_{x_0}^{\xi} I_k\right)$ of p independent variables $u_k = \sum_{i=1}^{p} \int_{a_i}^{x_i} I_k$ vanishes 1° when some of the points $\left(x_i, \overset{p}{\underset{1}{y_i}}\right)$ fall at (ξ, y_ξ); 2° when they are on an adjoint curve of the first kind, $\varphi(x^{n-2}, y^{n-2}) = 0$. If with Weierstrass we define this function by the equation

$$\theta\left(u_k \underset{1}{\overset{p}{-}} \int_{x_0}^{\xi} I_k\right) = e^{\int \Sigma\left[c_k + J\left(u_k \underset{1}{\overset{p}{-}} I_k\right)_k\right] du_k}, \tag{1}$$

where

$$J\left(\underset{1}{\overset{p}{u_k}}\right)_k = \sum_{i=1}^{p} \int_{x_0}^{x_i} \Pi_k \tag{2}$$

($\int_{x_0}^{x_i} \Pi_k$ denoting the integral of the second species, which becomes infinite when (x_i, y_i) falls at (a_k, b_k)), this property of θ must be derived from those of the function (2). For this purpose, considering in the first place the function

$$J\left(u_k \underset{1}{\overset{p}{+}} \int_{\xi}^{x'} I_k\right)_k = \sum_{i=1}^{p} \int_{x_0}^{a_i} \Pi_k \tag{3}$$

((x', y') denoting a point very near to (a_k, b_k)), where the points (x', y'), $\left(x_i, \overset{p}{\underset{1}{y_i}}\right)$ are the infinities, and (ξ, y_ξ), $\left(a_i, \overset{p}{\underset{1}{y_{a_i}}}\right)$ the zeros, of the principal function of (z, y_z)

$$P_{x,\xi}\left(x', y'; x_i, \overset{p}{\underset{1}{y_i}}\right) = \frac{P_{x',x_p}\left(z, y; x_i', \overset{p-1}{\underset{1}{y_i'}}; \xi, y_\xi \mid a_i, \overset{p}{\underset{1}{y_{a_i}}}\right)}{\varphi\left(z, \overset{n-2}{\underset{}{}} y_z^{n-2}; x_i, \overset{p-1}{\underset{1}{y_i}} \mid x_i', y_i'\right)} \tag{4}$$

we see, by the second form of this function, that in the two cases one of the infinities of the function in the numerator being absorbed by one of its arbitrary zeros (1° (x_p, y_p) by (ξ, y_ξ); 2° (x_p, y_p) by (x'_{p-1}, y'_{p-1})), the other will be absorbed by one of its non-arbitrary zeros $\left(\begin{smallmatrix} a_i, & y_{a_i} \\ 1, & p \end{smallmatrix}\right)$. Hence in the two cases, at the limit, for $(x', y') = (a_k, b_k)$, one of the integrals in (3) will become infinite like $\frac{1}{x - a_k}$ for $x = a_k$; thus θ will vanish.

More general interest was taken in M. Mittag-Leffler's papers, which followed. "Sur fonction analytique et expression analytique." "Une application de la théorie des séries n-fois infinies." "Sur une extension de la série de Taylor." In these M. Mittag-Leffler reported on his recent researches* in the theory of functions. Let $f(a)$, $f'(a)$, $f''(a), \cdots$ determine an element $P(x|a) = \sum_{n=0}^{\infty} \frac{1}{n!} f^n(a) \cdot (x - a)^n$ of an analytic function $f(x)$. This analytic expression is valid only for its circular domain of convergence; for a more extended region $P(x|a)$ must be supplemented by certain of its continuations. Let a lie within a continuous region which does not overlap itself, and let the branch of $f(x)$ derived from $P(x|a)$ by continuation throughout K be one-valued and regular; is it possible to find a *single* analytic expression for this branch $fK(x)$, where K is given its maximum extension, the formula to be based only on the primary quantities $f(a)$, $f'(a)$, $\cdots$? M. Mittag-Leffler has shown that such an expression can be built up in a comparatively simple manner. In doing this he has employed a new geometrical notion, that of the *star* (étoile). If a ray revolves about a, proceeding in each position to the nearest singular point of $f(x)$, this point lying it may be at infinity, the collection of points on the totality of such rays forms for $f(x)$ the star A; it is assumed that the lower limit of these rays is not zero. For A we have the theorem that

$$fA(x) = \sum_{m=0}^{\infty} G_m(x),$$

where $G_m(x)$ denotes a polynomial $\Sigma c_{n,m} f^n(a)(x - a)^n$, in

* "Sur la représentation analytique d'une branche uniforme d'une fonction monogène;" *Acta Math.*, vols. 23, 24 (1899, 1900).

which the coefficients $c_{n,m}$ are given initially and do not depend on a or on $f(a), f'(a), \cdots$.

The expression

$$G_n(x \mid a)$$

$$= \sum_{\lambda_1=0}^{n^2} \sum_{\lambda_2=0}^{n^4} \cdots \sum_{\lambda_n=0}^{n^{2n}} \frac{1}{\lambda_1! \cdots \lambda_n!} f^{(\lambda_1+\cdots+\lambda_n)}(a) \left(\frac{x-a}{n}\right)^{\lambda_1+\cdots+\lambda_n}$$

leads to a limiting expression $\lim_{n=\infty} G_n(x \mid a)$ with the following properties: It is uniformly convergent for every region interior to the star A, but never uniformly convergent for a region containing a vertex of A. Within A it represents the branch $fA(x)$ of $f(x)$.

It is perfectly possible that $\lim_{n=\infty} G_n(x \mid a)$ may converge outside A; the star A is not a star of convergence for $\lim_{n=\infty} G_n(x \mid a)$. M. Mittag-Leffler has shown that it is possible to replace $\lim_{n=\infty} G_n(x \mid a)$ by another expression for which A is a star of convergence. $\operatorname{Lim}_{n=\infty} G_n(x \mid a)$ was obtained from an n-fold series in x by making the maximum values of $\lambda_1, \lambda_2, \cdots, \lambda_n$ proceed simultaneously to the limit ∞. If the passage to the limit is performed in another way, viz., by taking $\sum_{\lambda_1=0}^{\lambda_1} \sum_{\lambda_2=0}^{\lambda_2} \cdots \sum_{\lambda_n=0}^{\lambda_n}$ in place of $\sum_{\lambda_1=0}^{n^2} \sum_{\lambda_2=0}^{n^4} \cdots \sum_{\lambda_n=0}^{n^{2n}}$ and then making $\lambda_n, \lambda_{n-1}, \cdots, \lambda_1$ tend *successively*, in the order named, to infinity, the expression

$$S_n(x \mid a) = \sum_{\lambda_1=0}^{\lambda_1} \sum_{\lambda_2=0}^{\lambda_2} \cdots \sum_{\lambda_n=0}^{\lambda_n} c_{\lambda_1 \cdots \lambda_n} f^{\lambda_1+\cdots+\lambda_n}(a) \left(\frac{x-a}{n}\right)^{\lambda_1+\cdots+\lambda_n}$$

(where the c's are given numerical constants of which

$$c_{\lambda_1} = \frac{1}{\lambda_1!}, \quad c_{\lambda_1\lambda_2} = \frac{1}{\lambda_1!\lambda_2!}\left(\frac{1}{2}\right)^{\lambda_1+\lambda_2}, \quad c_{\lambda_1\lambda_2\lambda_3} = \frac{1}{\lambda_1!\lambda_2!\lambda_3!}\left(\frac{1}{3}\right)^{\lambda_1+\lambda_2+\lambda_3},$$

while for values of $n > 3$ they are algebraic irrationals) yields the desired result; $\lim_{n=\infty} S_n(x \mid a)$ has the star A as a star of convergence, and represents $fA(x)$ within A. Writing $n = 1$, the series is seen to be simply Taylor's series; in general it is an extension of Taylor's series.

In the course of his remarks M. Mittag-Leffler referred to recent researches of M. Borel; this led to a discussion in

which MM. Borel, Hadamard, and Painlevé took part, on the nature of the connection between "analytic expression in a complex variable x" and "analytic function in x."

In Section III papers were presented by MM. Lovett, "On contact transformations between the essential elements of space"; Macfarlane, "Applications of space analysis to curvilinear coördinates"; Stringham, "Orthogonal transformations in elliptic or in hyperbolic space"; Amodeo, and others. In Section IV very few papers were read; one appointed meeting of the section was not held, and some of the papers intended for the section were presented at the joint sitting of Sections V and VI, which was transformed momentarily into a sitting of Section IV, to hear MM. Hadamard: "Relations entre les caractéristiques réelles et les caractéristiques imaginaires pour les équations différentielles à plusieurs variables indépendantes"; and Volterra: "Comment on passe de l'équation de Poisson à caractéristique imaginaire à une équation semblable à caractéristique réelle."

The communications made in Sections V and VI, while not necessarily the most valuable mathematically, were yet of the most general interest, and lend themselves best to any general report. The sitting was opened by M. Hilbert's address, in German, on the future problems of mathematics. The lines along which we may expect the development of any science which is progressing in a continuous manner can be detected by an examination of the problems to which attention is specially paid. Among these most importance is to be attached to those that are sharply defined and stand out well; such, for example, as the problem of three bodies. In the earlier stages of any science, problems present themselves naturally through experience, as is exemplified in mathematics by the duplication of the cube and the quadrature of the circle, and at a later date by the questions arising with reference to infinitesimal analysis and the theory of the potential; but as the science progresses, it is the logical faculty of the intellect that imposes on us problems such as are found in the theories of prime numbers, elliptic functions, etc.

As to our aim with regard to any problem, there must be a definite result of some kind, it cannot be laid aside until we have obtained either a satisfactory solution or a rigorous demonstration of the impossibility of a solution. The mathematical rigor that is essential in the treatment of a

problem does not require complicated demonstrations; it requires only that the result be obtained by a finite number of logical steps from a finite number of hypotheses furnished by the problem itself; in seeking this rigor we may find simplicity. The proper treatment of any problem depends on 1° a complete system of axioms, by means of which the conceptions are defined, 2° a system of symbols appropriate to the conceptions with which the problem deals; thus a demonstration by means of geometrical symbols is as legitimate as an arithmetical one, provided that the axioms on which it is based are perfectly understood. The mere formulation of these axioms is in some cases itself the problem, as for instance in arithmetic and physics. Among the ten problems that M. Hilbert specified in particular as fitted to advance mathematics, No. 2 is that of finding some one system of independent compatible axioms governing and defining arithmetical conceptions, and No. 3 is the same question for the calculus of probabilities, rational mechanics, and physics. Other problems are to prove that $e^{i\pi z}$ is transcendental when z is an algebraic irrational; and that the solution of the general equation of the 7th degree cannot be obtained by means of a finite number of operations involving only two parameters. In geometry, the relative situation of the circuits that a plane curve of assigned order can possess, with the corresponding question as regards surfaces, demands investigation; in the theory of functions there is the question of the expression of two variables, connected by any analytic relation whatever, as uniform functions of a single parameter z—for Poincaré's theorem (*Bulletin de la Société mathématique de France*, volume 11 (1883)) is subject to some limitations. These are but a few of the problems that M. Hilbert mentioned, and these were a selection from a much longer list for which he referred to an article about to appear in the *Nachrichten der Kgl. Gesellschaft der Wissenschaften zu Göttingen*, 1900. In the course of a rather desultory discussion that followed the reading of this paper, the claim was made, though apparently without adequate grounds, that more had been done as regards the equation of the 7th degree (by some German writer) than the author of the paper was willing to allow. A more precise objection was taken to M. Hilbert's remarks on the axioms of arithmetic by M. Peano, who claimed that such a system as that specified as desirable has already been established by his compatriots MM. Burali-Forti, Padoa, Pieri, in memoirs referred to on pp. 3–5 of no. 1, volume 7 of the *Rivista di Matematica*.

M. Hilbert was followed by M. Fujisawa, the official delegate from Japan, who gave, in English, a very interesting account of the mathematics of the older Japanese school. It is difficult to follow the course of Japanese mathematics; there are some two thousand manuscript volumes still to be transcribed; in these much valuable work is mixed up with what is purely elementary and even trivial. The difficulty of arriving at any clear idea is greatly increased by the fact that publication of results was not customary; they were preserved to a great extent only by oral transmission. So far as the books have been deciphered and collated, one fact stands out with ever-increasing clearness, and that is that side by side with one less important school of Japanese mathematics there exists another earlier system of mathematics of a peculiar kind, which had its origin in Japan, and was developed there entirely free from any external influences.

The mathematics of the first kind, probably derived from the Chinese at a very early date, displays a noticeable lack of rigor; for instance, $\sqrt{10}$ is accepted as the value of π. As to content; in algebra, the solution of simple equations and the formulæ for the sum of an arithmetical or geometrical progression were known; in geometry, the right-angled triangle with sides proportional to 3, 4, 5 was used, with some propositions regarding regular polygons; magic squares were discussed, even so far as those containing the first 400 numbers. Bamboo rods were used for purposes of calculation; these were placed one above another to indicate addition, side by side to indicate multiplication, diagonally to denote subtraction.

The other part of Japanese mathematics, that indigenous to the country, is of more importance and interest. It appears that the mathematicians of this school made use of local value in expressing numbers, invented zero for themselves, and used the circle as the symbol for zero. They were familiar with imaginaries and complex numbers; and were such adepts at calculation that they found the value of π correct to 49 places of decimals. M. Fujisawa explained that the knowledge of this part of Japanese mathematics so far obtained is very fragmentary, the unexplored part offers an attractive field of research for Japanese who may care to devote themselves to it. It is a matter of purely historical interest, as the present teaching of mathematics in Japan is in no sense founded on it; for, very wisely as he thinks, the Japanese educational authorities made an entirely fresh start, sweeping away all trace of this older educational system.

The president of the section, M. Cantor, then spoke of the difficulties he encountered, when writing his Geschichte der Mathematik, in finding out anything about the earlier Japanese mathematics. When he did finally hear of a work of reference it turned out to be written in Japanese. With reference to the earliest use of zero, he expressed the opinion that it was probably due to the Babylonians, about 1700 B. C.

Another paper of interest in these sections was that of M. Padoa (Rome) on Friday morning: "Un nouveau système irréductible de postulats pour l'algèbre." Naming the object, *entier* (integer), two undefined derivatives, *successif* and *symétrique*, are considered. The seven postulates are

1°. If a is an integer, then suc. a is an integer.

2°. If a is an integer, then sym. a. is an integer.

3°. If a is an integer, then sym. (sym. a) $= a$.

4°. If a is an integer, then sym. $\{$suc. [sym. (suc. a)]$\} = a$.

5°. There exists an integer x such that sym. $x = x$.

6°. There do not exist two different integers x, y, such that sym. $x = x$, and sym. $y = y$.

7°. If a class u of objects satisfies the conditions

(i) it contains some one integer,

(ii) if it contains an integer x it contains also suc. x,

(iii) if it contains suc. x it contains also x,

then every integer belongs to the class u.

These postulates define an algebraic field, whose nature is at once seen to agree with that of the natural field, suc. x being interpreted as $1 + x$, and sym. x as $-x$. M. Padoa did not get beyond this definition, possibly because he had entered so minutely into the details of the proof of the independence of the seven postulates that he had exhausted his allowance of time.

A great part of the Friday morning sitting of these two sections was devoted to the discussion of a resolution, proposed by M. Leau, in favor of the adoption of some special artificial language as the vehicle for all scientific communications. Though no particular language was named in the resolution, it was made clear that "Esperanto" was the language intended. Its advocates, MM. Leau, Padoa, Boccardi, Laisant, and others, disclaimed any wish to substitute it for natural languages, but urged its adoption as the vehicle for international intercourse; this view they upheld with great earnestness "on behalf of humanity," as M. Laisant put it. The opposite view was upheld with equal earnestness, if less vehemence, by MM. Schroeder, Vassi-

lief, Maggi, and others, chiefly on the ground that any such language is entirely unnecessary; as M. Maggi remarked, mathematics already has a universal language, the language of formulæ. In the end the suggestion of M. Vassilief was adopted, that the Congress should place itself on record as opposed to any unnecessary diversity in the languages employed, that is, practically, to the use of any language for scientific purposes other than English, French, German, and Italian, though these languages were not specified in the resolution adopted.

On Saturday, August 11th, the concluding general meeting was held at 9 a. m. The first business was to determine the time and place for the next meeting. At Zürich, Professor Klein, on behalf of the German Mathematical Society, had expressed their great desire that the third congress should be held in Germany; and a definite invitation to this effect was now laid before the Congress, and unanimously accepted. The place of meeting will probably be Baden-Baden; the date decided upon is 1904, and the time is to be either at the beginning or the end of the summer vacation. No other business was transacted, and the two general addresses appointed for the day were then delivered by MM. Mittag-Leffler and Poincaré. Immediately after the conclusion of the President's address, he dismissed the Congress with the words "La séance est levée, le congrès est clos."

M. Mittag-Leffler's address was entitled "Une page de la vie de Weierstrass"; in this he considered in some detail Weierstrass's attitude towards some of the mathematical ideas of his time, illustrating by copious extracts from his correspondence; unfortunately it is not possible to give any adequate account of it. M. Poincaré's can be given more fully.

H. POINCARÉ: *Du rôle de l'intuition et de la logique en mathématiques.*

It is obvious that there are two entirely different types of mind among mathematicians, manifesting themselves in two distinct methods of treating mathematical questions. Those of the first type are dominated by logic; those of the second are guided by intuition. They may be called analysts and geometers, though it is not really a question of the subject with which they deal; the analyst remains an analyst even when working at geometry, and the geometer

employing himself on pure analysis is still a geometer. Nor is the distinction a mere matter of education; a man is born a mathematician, he does not become one; and either he is born an analyst or he is born a geometer. The two types of mind are equally necessary for the progress of the science; each has accomplished great things that would have been impossible to the other.

At first sight the ancients seem to have all been intuitionalists, but this impression disappears on closer study. Euclid, for instance, was a logician, even though every stone of his edifice is due to intuition. The natural tendencies have not changed, only their manifestation. There has been an evolution, due to the increasing recognition of the fact that intuition cannot give rigor, nor even certainty; a proof that relies on concrete images may be very deceptive. It was soon realized that rigor cannot be expected in the demonstrations unless it is to be found in the definitions; so long as the objects of reasoning were given simply by the bodily senses or the imagination, there was no precise idea on which reasoning could be based. Thus the efforts of the logicians were concentrated on the definitions, one result of which is that mathematics has become arithmetized.

The question arises, is this evolution ended—have we at last attained to absolute rigor, or do we deceive ourselves as our fathers did? Philosophers tell us that it is impossible to eliminate intuition altogether from our reasonings, for no science can spring from pure logic alone. To designate this other essential, we have no name but intuition; but this covers many different ideas. There is (1) the appeal to the bodily senses and to imagination; (2) generalization by induction; (3) the intuition of pure number; on this last a veritable mathematical method is based, while from the first two no certainty can be derived. The analysis of the present day constructs its demonstrations solely from syllogisms and this intuition of pure number; we may say that at last absolute rigor is attained.

The philosophers now object that what has been gained in rigor has been lost in actuality; the approach toward the logical ideal has been secured by cutting the ties with reality. For the sake of the demonstration a mathematical definition is substituted for the object, and it still remains to prove that the concrete reality answers to the definition. But as this is an experimental truth, it is not the business of mathematics to establish it. It is a great step forward to have separated these two things; nevertheless there is some-

thing in the philosophic objection. In becoming rigorous, mathematics has assumed a certain character of artificiality; if it is clear how questions can be resolved, it is no longer clear how and why they arise. We seek for reality; but this does not reside in the separate steps of the demonstration; it must be sought rather in the something that makes for unity. The microscopic examination of an elephant gives no idea of the animal itself; the fairy-like structure of silicious needles which is all that is left of certain sponges cannot be understood without reference to the living sponge by which this form was imposed on the silicious particles. Logic by itself cannot give the view of the whole which is indispensable alike to the inventor and to him who desires really to understand the inventor. Logic, which alone gives certainty, is simply the instrument of demonstration; the instrument of discovery is intuition.

But analysts also are inventors; hence they cannot always be proceeding from the general to the particular, as the rules of formal logic demand, for scientific conquests are made only by generalization. There is however a perfectly rigorous process, that of mathematical induction, by which it is possible to pass from the particular to the general.* For the profitable use of this, to recognize the analogies whose presence makes it applicable, the analyst must have the direct feeling for the unity of an argument, for its soul and spirit; for him the most abstract entities must be living beings. What is this but intuition? This however does not invalidate the distinction already drawn, for it is an intuition entirely different in nature from the sensible intuition founded in imagination alone, even though psychologists may finally pronounce it also to have a sensual foundation. It is the intuition of pure logical form, which together with the intuition of pure number makes not only demonstration, but also discovery, possible to the analyst. Thus among the analysts inventors do exist, but not many; it remains true that the most usual instrument of invention in mathematics is sensible intuition.

On the evening preceding the formal opening of the Congress an informal reunion of the members, about half of whom were present, was held at the Café Voltaire. On Tuesday afternoon, after the rising of the sections, the members were entertained at the École normale supérieure. At noon on Sunday, August 12th, a very successful banquet

* Poincaré, "Sur la nature du raisonnement mathématique," *Revue de métaphysique et de morale*, vol. 2 (1894), pp. 371–384.

was held at the Salle de l'Athénée-Saint-Germain, when, in the absence of M. Poincaré, M. Darboux presided very pleasantly. Toasts to those present, to the hosts, to the absent, to M. Darboux, and to the next Congress, were proposed by MM. Darboux, Geiser, J. Tannery, Stephanos, and Vassilief. Invitations to receptions held by the President of the Republic and by Prince Roland Bonaparte were accepted by a number of the members.

A very large attendance had been expected, on account of the additional attractions offered by the Exhibition; and the answers to the circulars first sent out went far to justify this anticipation, for up to last December about 1,000 mathematicians had signified their intention of being present, with 680 members of their families. The membership fee was fixed at 30 francs, with an additional 5 francs for every member of the family. As a matter of fact, the total attendance can hardly have exceeded 250 in all. There seems very little doubt that a large proportion were kept away by distaste of the crowds that were supposed to be visiting the Exhibition, and by the rumored difficulty in obtaining accommodation, a difficulty that seems to have existed mainly in the circulars of the various agencies; but the great heat of July certainly decided many to stay away who would otherwise have been present.

The countries represented were as follows: France, 90; Germany, 25; United States, 17; Italy, 15; Belgium, 13; Russia, 9; Austria, 8; Switzerland, 8; England, 7; Sweden, 7; Denmark, 4; Holland, 3; Spain, 3; Roumania, 3; Servia, 2; Portugal, 2; South America, 4; with single representatives from Turkey, Greece, Norway, Canada, Japan, Mexico. This list is only approximate, as no revised list of members was issued. Among the members from the United States were Professors Allardice, E. W. Brown, Dickson, Ely, Hagen, Halsted, Hancock, Harkness, Keppel, Lovett, Macfarlane, Pell, Scott, Stringham, Webster.

While the four languages, English, French, German, and Italian, were admitted on equal terms, by the constitution of the congresses, the great preponderance of French was noticeable. At Zürich, this preponderance existed in friendly intercourse, but French and German were about equally used in the communications, whereas in Paris all the general addresses, and most of the sectional papers, were in French, possibly out of compliment to our hosts. Probably at Baden-Baden French and German will be used in about equal proportions as at Zürich.

The distribution of the authors of communications among

the different countries may be of interest. The four general addresses were delivered by representatives from France, Italy, Germany, Sweden; including these, papers were read by members from France, 13; Italy, 9; United States, 6; Germany, 4; Sweden, 4; Austria, 2; England, Greece, Holland, Japan, Portugal, Russia, Servia, and Spain, one each. This list may require some slight modification, as the only programme issued needed some corrections, but it is substantially correct. In some cases two or three communications were made by one member, thus bringing up the number of sectional papers to about 50; some of these were presented by title only in the absence of their authors. The general meetings occupied four hours, the sectional meetings 26. About 200 members were present at each of the general meetings; at the sectional meetings the average attendance was about 90, and as two sections were usually sitting at the same time, this accounts very fairly for the members. About 160 were present at the banquet.

The arrangements excited a good deal of criticism. The committee of organization had doubtless special difficulties to contend with, as M. Laisant, to whom the secretarial part had been assigned, was unable to undertake it owing to the pressure of other duties. The mistake was then made of entrusting a part of this responsibility to the firm of Carré and Naud, whereas in such a case personal interest and individual responsibility are indispensable for ensuring proper attention to the various details of organization. Owing to this, members arriving in Paris had very great difficulty, during the first two days, in obtaining the necessary information in time for it to be of any service. The want of a common assembly room, where members might conveniently meet one another with or without concerted arrangement, was seriously felt. The arrangements in Zürich were so admirably complete in every point, that these defects were even more conspicuous by comparison. There is no doubt that a smaller town lends itself best to such a gathering; it is not so much that there is less division of interests, as that the members are more in evidence, and so have a better chance of *realizing* one another. The first object of an international congress of mathematicians is to enable its members to meet one another in circumstances that shall excite the mathematical faculty, and make manifestations of mathematical interest most natural, thus encouraging the exchange of ideas both between individuals and larger groups. The mathematician

who is in any degree a specialist is in general rather solitary in the average college—he would have been better off in Noah's Ark, for at the worst there would have been two of a kind. For his mind's health it is well that he should occasionally be thrown with those of kindred interests; it is well too that he should be made to feel the unity of mathematics.

The general addresses are of course under one aspect the most important part of the formal proceedings, giving broad views of the whole subject, and helping the pronounced specialist to realize his affiliations with other regions. Did a congress achieve no other end than that of evoking such addresses as those of Klein, Hurwitz, Cantor, Poincaré, Volterra, at Zürich and Paris, and presenting them to mathematicians of such widely differing interests, its existence would be amply justified. Something more might be done in a similar direction as regards the sectional meetings; an address, not necessarily by the president of the section, dealing in a broad manner with the region, or some part of the region, assigned to the section, might very well be arranged for. While the general addresses enable the individual to appreciate the relation of his special subject to mathematics as a whole, these sectional addresses would assist him in the equally important and even more difficult task of gauging his own relation to his special subject. One or two of the communications offered, both at Paris and at Zürich, were of this nature; but the matter should not be left to chance, it is well worth systematically arranging. Such an address would deal sometimes with the general ideas of the subject, sometimes with what remained to be done, sometimes with what had been accomplished during the last few years.

As to the nature of the more special papers, it hardly seems advisable to restrict the present liberty, not even to the extent that is found salutary in the regular meetings of a mathematical society; encouragement is perhaps more needed. But some control over the time consumed should be exercised; not only a theoretical control, confined to a printed statement that it must not exceed twenty minutes, but a real control, exerted as a regular thing. Ten or fifteen minutes, well employed, is quite long enough for the ordinary type of special theorem, for an outline of the method of proof is sufficient in an oral communication, and another five or ten minutes ought to enable the speaker to indicate its connections within the subject. The strict limitation to a time previously determined would in most

cases be beneficial to the author, obliging him to select, subordinate, and group his details. It is tolerably certain that if the author regards all details as equally important, his auditors will regard all as equally unimportant.

One thing very forcibly impressed on the listener is that the presentation of papers is usually shockingly bad. Presumably the reader desires to be heard and understood; to compass these ends, instead of speaking to the audience, he reads his paper to himself in a monotone that is sometimes hurried, sometimes hesitating, and frequently bored. He does not even take pains to pronounce his own language clearly, but slurs or exaggerates its characteristics, so that he is often both tedious and incomprehensible. These failings are not confined to any one nationality; on the whole the Italians, with their clear and spirited enunciation, come nearest to being free from them. It would be invidious and impertinent to mention names; the special sinners sit in both high and low places. But it is perhaps pardonable to refer to M. Mittag-Leffler's presentation of his paper to Section II as showing in how admirable and engaging a style the thing can be done. It is not given to everyone to do it with this charm; but there is no excuse for any normally constituted human being, sufficiently versed in mathematics, failing to interest a suitable audience for a reasonable time in that which interests himself, always provided that it be of sufficient novelty either in matter or in mode of treatment to justify him in presenting it at all.

At the Zürich Congress certain matters were energetically discussed in Section V; extensive support was then given to resolutions in favor of constituting permanent commissions charged to consider 1° general reports on the progress of mathematics, 2° matters of bibliography and terminology, 3° the possibility of giving some permanent character to the Congress, by means of a central bureau or otherwise. Though these resolutions were not voted upon directly, it being felt that they required more deliberate discussion, yet at the concluding general meeting the members of the Zürich bureau were appointed a commission to consider the questions that seemed of most importance, and to furnish the Mathematical Society of France with such information on these points as might be useful in preparation for the Congress of 1900. At the joint sitting of Sections V and VI in Paris M. Dickstein asked a question on behalf of the members, namely, was not the Congress to hear anything of the deliberations of this commission? No satisfactory answer was forthcoming; M. Laisant replied simply that

the Mathematical Society of France had been so taken up with material preparations for the Congress that it had not been able to enter upon any of these matters. He took the opportunity, however, of directing attention to the *Annuaire des Mathématiciens*, projected by Carré and Naud, as carrying into effect one suggestion made at Zürich. The question then dropped, but it was felt that this left matters in a very unsatisfactory state. It is to be hoped that a different report will be given in 1904, that the members of the Baden congress will not simply hear papers and meet friends, but have a chance to consider these matters of international concern. Some questions of business arise in every science; they tend to settle themselves by a kind of tentative process, a survival of the fittest, or rather by general tacit consent. This is often the best process; any attempt at forcing an expression of general agreement may result in checking development by encouraging too early a crystallization. But there are some matters, depending on concerted action, that are ripe for decision, and that cannot profitably be settled by any one nation for itself; matters in which for want of a general agreement labor may be wasted. Such matters may naturally be decided by an international congress, whose decisions will simply have the force of general consent. One such question is that of a classification of mathematical sciences. At least two well-known systems are in use, and there may be others. Multiplication of systems of classification, like the multiplication of universal languages, practically destroys the good of any and all; the congress ought to pronounce in favor of some one. As to the preparation of special reports, it seems doubtful whether this will be done best by the congress at present. In course of time it may assume academic functions and responsibilities, but it will be necessary for it to prove its continuity before it can with any propriety expect to control mathematical efforts. Encouragement and recognition would seem to be its appropriate province at present in these respects. The question as to how this continuity is to be obtained is a rather serious one, and deserving of careful discussion. A central bureau with various functions was suggested when the matter was under discussion at Zürich; but there are objections to this. If the organizers of each congress will make it a point of honor to act on the recommendations of the preceding congress, taking into consideration the resolutions passed and doing what can be done towards carrying them into effect, possibly sufficient continuity may be attained without the red tape that would coil itself about any

permanent bureau. Some questions are better left undecided. International agreement is not wanted on all points; international rivalry and emulation still have their part to play, helped by the international friendships that are promoted by such gatherings as these international congresses.

In conclusion, I must express my thanks to many of those whose names appear in this report for the assistance I have received from them in its preparation.

CHARLOTTE ANGAS SCOTT.

BRYN MAWR COLLEGE,
October, 1900.

THE FORTY-NINTH ANNUAL MEETING OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

THE forty-ninth annual meeting of the American Association for the Advancement of Science, which was held at Columbia University June 23–30, was from the point of view of scientific work one of the most successful in the history of the Association. Sixteen affiliated societies met with the Association and contributed greatly to the importance and interest of the meeting. Two of these, the American Mathematical Society and the Astronomical and Astrophysical Society, held joint sessions with Section A.

The next meeting of the Association will be held in Denver during the last week of August, 1901, under the presidency of Professor Minot of the Harvard Medical School. Professor James McMahon, Cornell University, will be vice-president of Section A, and Professor H. C. Lord, Ohio State University, will be secretary. Forty-one out of the total number of forty-nine annual meetings of the Association have been held during the month of August, while the recent New York meeting was the first that was held in June. The next meeting will be farther west than hitherto, but it seemed to be the general opinion that this was desirable in order to extend the influence of the Association. Pittsburg was recommended as the place of meeting in 1902.

The meetings of the section of mathematics and astronomy were well attended. The officers of this section were: vice-president, Asaph Hall, Jr.; secretary, W. M. Strong; coun-

cilor, Ormond Stone; sectional committee, Alexander MacFarlane, J. F. Hayford, C. L. Doolittle, W. M. Strong, Harold Jacoby, Edgar Frisby, W. S. Eichelberger. The Council elected the following mathematicians and astronomers to fellowship in the Association: E. S. Crawley, G. A. Hill, W. J. Humphreys, H. C. Lord, L. H. Orleman, Mary W. Whitney, P. S. Yendell.

The opening address of the Vice-President of Section A, "On the teaching of astronomy in the United States," has been published in *Science*, July 6, 1900, pp. 15–20. In the absence of the Vice-President, Simon Newcomb, C. L. Doolittle and Ormond Stone presided at the meetings of this Section. The following list of papers read before the Section does not include those of the American Mathematical Society and of the Astronomical Society which were read at the joint sessions of these societies and Section A.

(1) President H. S. PRITCHETT: "The functions, organization, and future work of the U. S. Coast and geodetic survey."

(2) Mr. J. F. HAYFORD: "The precise level net of the U. S. and a new levelling instrument."

(3) Professor T. J. J. SEE: "The propagation of the tide wave."

(4) Professor T. J. J. SEE: "The dimensions and density of Neptune."

(5) Professor E. FRISBY: "Some remarkable properties of recurring decimals."

(6) Professor G. T. SELLEW: "History of the complex number."

(7) Professor JAMES MCMAHON: "The expression of a rational polynomial in a series of Bessel Functions of the nth order."

(8) Dr. G. A. MILLER: "Report on the groups of an infinite order."

(9) Professor L. E. DICKSON: "Definition and examples of Galois fields."

(10) Mr. W. B. FITE: "On the metabelian groups whose invariant operators form a cyclical subgroup."

(11) Mr. H. E. HAWKES: "Note on Benjamin Pierce's linear associative algebra."

(12) Mr. F. H. LOUD: "Sundry metrical theorems connected with a special curve of the fourth class."

(13) Mr. G. A. HILL: "Variations of latitude."

(14) Mr. G. A. HILL: "The comparative accuracy of the transit circle and the vertical circle."

(15) Professor J. N. STOCKWELL: "New light on ancient eclipses."

(16) Professor J. N. STOCKWELL: "Secular variations of the motions of the planets."

(17) Professor J. N. STOCKWELL: "A new method of computing the Laplace coefficients of planetary pertubation."

(18) Miss MARY PROCTOR: "Miss Catharine Wolf Bruce."

(19) Miss M. E. TRUEBLOOD: "The directive force of philosophy upon mathematics."

(20) Mr. W. G. LEVISON: "On a method of photographing the entire corona, employed at Newberry, S. C., for the total solar eclipse, May 28, 1900."

Two of the titles that appeared on the programme are not reproduced here since these papers did not reach the secretary in time to be presented at the meeting. In the absence of their authors the papers by Professor McMahon and Mr. Fite were presented by Professor Crawley and Dr. Miller respectively. Mr. Hawkes' paper was read by title. The two papers by Mr. Hill will be published in the *American Journal of Science* and in *Science* respectively. Dr. Miller's report will appear in the BULLETIN. Abstracts of the other papers, with the exception of the last three, are given below.

President H. S. Pritchett, lately Superintendent of the U. S. Coast and geodetic survey, considered the three questions: "What is the purpose of this service? Is it properly organized to carry out this purpose? What lines of work should be followed to accomplish the purpose in view?" The first and principal work of the Coast and geodetic survey is the hydrographic survey of the coasts of the United States and the islands under her jurisdiction. In hydrography and topography a part of the energy of the service will go to the resurvey of parts of the mainland of the United States; but by far the greater effort will go to the hydrographic surveys of Porto Rico, Hawaii, the Philippine Islands. and Alaska. In geodesy, the most interesting work of the next eight years will be the completion of an arc extending along the ninety-eighth meridian from the Rio Grande to the Canadian border. By the coöperation of Mexico and of Canada it is expected that this arc will ultimately have for its southern limit a latitude of about 19 degrees and for its northern terminus the latitude of nearly 70 degrees. The completion of the precise level net for the United States, and the differential gravity determinations now being carried out in connection with similar work by

other nations, are the most important other objects of this service. The work of the Division of terrestrial magnetism contemplates a general magnetic survey of the country and of the waters adjacent within the next ten years.

Mr. Hayford's paper is in abstract as follows: The net of precise level lines recently adjusted in the Computing Division of the Coast and geodetic survey comprises over 12,000 miles of spirit leveling, and 1,400 miles of water leveling, run by the Coast and Geodetic Survey and other organizations. The manner of making several preliminary "studies" before making the final least square adjustment was explained. The points requiring especial attention were, the assignment of relative weights to different classes of leveling, the attempt to detect systematic errors, and the assignment of the relation of weight to length of line. The least square method of computation was used throughout and the assumptions upon which it is based were checked, as far as possible, by direct appeals to the facts. The only well-marked form of a systematic error found, is one which is due to changes of temperature in the instrument during the period of observation at a station. The instrument shown was one which is about to be put into use in the Coast and geodetic survey; it is designed with special reference to eliminating the systematic error just referred too, and to furnish rapid observations of the highest degree of accuracy.

The object of the first paper by Professor See, was to direct attention to certain points in the theory of the propagation of the tide wave. After defining the tide wave by the general relations for horizontal and vertical motion,

$$x' = x + \varphi(x,t), \quad y' = y + \psi(y,t),$$

which are transformed into

$$Z = A\left(e^{\frac{2\pi\gamma}{\lambda}} + e^{-\frac{2\pi\gamma}{\lambda}}\right)\cos(nt - \nu x),$$

$$H = -A\left(e^{\frac{2\pi\gamma}{\lambda}} - e^{-\frac{2\pi\gamma}{\lambda}}\right)\sin(nt - \nu x),$$

γ being the depth of the sea, λ the wave length, and ν any integral number, the author shows that in case of the tides, where λ is large compared to γ, the path of any particle is an elongated ellipse, giving a horizontal motion about 1,000

times greater than the vertical motion. He then gives the tide-generating potential in the usual form

$$V = \frac{3}{2}\frac{m\nu^2}{\rho^3}\left(\cos^2\omega - \frac{1}{3}\right),$$

and shows by derivation the lengths of the four arcs into which the circumference is divided, two being each 109°.5 in length, the other two 70°.5. A figure showing the motion of the water in every part of an equatorial canal at any instant, indicates clearly why the wave advances when once started, as the flow is forward in the two long arcs, and backward in the short arcs. The author states that the tide wave is at first a forced wave, afterwards acting as a free wave; but this view of the propagation holds for both classes of oscillations, when treated separately, and is approximately true when the two classes of waves are compounded.

In his second paper Professor See treats historically the question of the diameter of Neptune, showing that the earliest measures, made in 1846, place it at about 3″, and that subsequent measures range from 4″.3 to 2″.00. With every improvement in the definition of telescopes, the diameter has shrunk. He then showed, from observations made in 1899–1900 with the great equatorial of the Naval Observatory under better conditions than have been enjoyed by previous observers, that the true diameter of the planet is about 2″.00, or 27,190 miles. The diameter hitherto accepted in standard works on astronomy is 34,000 miles. The former density was 1.11 that of water, but this is now doubled and put at 2.26.

Professor Frisby gave some brief special methods of combining numbers, and especially of finding the recurring decimals which represent rational fractions.

Professor Sellew traced the history of the complex number from the first appearance of the imaginary in the Stereometrica of Hero of Alexandria in the second century B. C. The idea that the imaginary must be in the plane without the line of real numbers was first given by Wallis (1685). Kühn's construction (1753) introduced the idea of direction. Wessel (1797) was the real inventor of the modern representation of the complex number. His analytic representation of direction was the best before the time of Hamilton.

It is known that in certain cylindrical boundary problems it is necessary to expand $f(r, \theta)$ in a Fourier series of the form

$$\sum_{n=1}^{\infty} \varphi_n(r) \cos n\theta,$$

and then to expand $\varphi_n(r)$ in an nth order Bessel series of the form

$$A_1 J_n(\lambda_1 r) + A_2 J_n(\lambda_2 r) + \cdots,$$

in which $\lambda_1, \lambda_2, \cdots$ are determined from the roots of a certain equation to be satisfied on the boundary $r = a$, and the coefficients are given by

$$A_s = \frac{\int_0^a \varphi_n(r) . r J_n(\lambda_s r) dr}{\int_0^a r [J_n(\lambda_s r)]^2 dr}.$$

The value of the definite integral in the denominator is given in the text-books, but no method of evaluating the integral in the numerator has been given. The object of the paper by Professor McMahon is to reduce this integral in the commonest case, viz., that in which $\varphi_n(r)$ is expressible in a series of powers of r with integer exponents (positive or negative). For this purpose formulæ are given by means of which $\int x^m J_n(x) dx$ is made to depend on $\int x J_n(x) dx$ and $\int J_n(x) dx$, and ultimately on $\int J_0(x) dx$ alone.

Thus the complex numerical treatment of a large class of cylindrical harmonic problems requires the introduction and tabulation of a new fundamental transcendent, which may be defined by the equation

$$H_0(x) = \int_0^x J_0(x) dx.$$

This function may be tabulated either from the power-series, or from the relation

$$H_0(x) = 2[J_1(x) + J_3(x) + J_5(x) + \cdots].$$

Professor Dickson developed the definition of Galois field as follows: Let p be a prime number and $P(x)$ a polynomial of degree n in x having integral coefficients, and suppose $P(x)$ to be *irreducible* modulo p, *i. e.*, suppose it to be impossible to give $P(x)$ the form

$$P(x) = P_1(x) \cdot P_2(x) + p \cdot P_3(x),$$

where the $P_i(x)$ are polynomials having integral coefficients, the degrees of $P_1(x)$ and $P_2(x)$ being $<n$. If we divide any polynomial $F(x)$ having integral coefficients, by the function $P(x)$, we obtain a quotient $Q(x)$, and a remainder which can be written

$$f(x) + p \cdot q(x), \quad f(x) \equiv a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1},$$

where every a_j is a positive integer $< p$. Hence

$$F(x) \equiv f(x) + P(x)\,Q(x) + p \cdot q(x).$$

We say $f(x)$ is the residue of $F(x)$ modulo p and $P(x)$ and write

$$F(x) \equiv f(x) \quad [\text{mod } p, P(x)].$$

The totality of functions $F(x)$ obtained by giving to the polynomials $Q(x)$ and $q(x)$ all possible forms is said to constitute a class of residues. Two polynomials

$$F_i(x) = f_i(x) + P(x)\,Q_i(x) + p \cdot q_i(x) \quad (i = 1, 2)$$

belong to the same or different classes according as f_1 and f_2 are identical or not. Each a_j in $f(x)$ having p values, there are evidently p^n distinct classes of residues. The class to which $F_1 \pm F_2$ or $F_1 \cdot F_2$ belongs depend merely upon the functions $f_1 \pm f_2$ or $f_1 \cdot f_2$, being independent of the Q_i and q_i. Hence classes of residues combine unambiguously under addition, substraction and multiplication. Furthermore, an arbitrary class C_{F_2} may be divided by any class C_{F_1}, not the class zero, yielding as quotient a unique third class. In fact, p being prime and $P(x)$ irreducible modulo p, and $F_1(x)$ being not divisible by $P(x)$ modulo p, we can determine, by a process analogous to that for finding the greatest common divisor, integral functions $F_1'(x)$ and $P'(x)$ such that

$$F_1'(x) \cdot F_1(x) - P(x) \cdot P'(x) \equiv 1 \pmod{p}.$$

Hence the function $F_1' F_1$ belongs to the class unity. It follows that

$$\frac{C_F}{C} = \frac{C_{F_2} C_{F_1'}}{C_{F_1} C_{F_1'}} = C_{F_2 F_1'}.$$

The p^n classes of residues modulis p and $P(x)$ therefore form a field, called a Galois field of order p^n.

The main object of Mr. Fite's paper was to demonstrate the following theorems:

THEOREM I. If the commutators of a metabelian* group G of order p^m, p being a prime, form a cylical group of order p^α, then the number of independent generators of any given order of the group of cogredient isomorphisms G' must be even.

THEOREM II. There are exactly $\beta + 1$ groups of order $p^{\alpha+2\beta s}$, p being an odd prime, s being any positive integer, and $\beta \leqq \alpha$, such that their invariant operators form a cylical group of order p^α and that their groups of cogredient isomorphisms are abelian of the type (β, β, $\cdots$ to $2s$ terms).

THEOREM III. The number of metabelian groups whose invariant operators form a cyclical group of order p^α and whose groups of cogredient isomorphisms are of the type (a_1, a_1, $\cdots$, to $2s_1$ terms; a_2, a_2 $\cdots$ to $2s_2$ terms; $\cdots$; a_n, a_n, $\cdots$ to $2s_n$ terms), where s_1, s_2, $\cdots$, s_n are positive integers and $a_1 < a_2 < a_3 < \cdots < a_{n-1} < a_n$, is $(a_1+1)(a_2-a_1+1)(a_3-a_2+1)$

$$\cdots(a_n-a_{n-1}+1).$$

The curve studied by Mr. Loud is the parallel to the hypocycloid of four cusps, and is briefly noticed by Salmon in his Higher plane curves, §§ 118 and 119. The present paper gives a simple method of construction by points and establishes several theorems connecting the curve with the metric geometry of n lines in a plane. The treatment adopted employs the circular coördinates x and y, equal respectively to $x + iy$ and $x - iy$ in the Cartesian system. Some of the more obvious properties of the curve, as well as its varieties of form, are deduced. The methods and results given by Professor Morley in the second number of the *Transactions* have been extensively employed.

In his first paper Professor Stockwell argues that ancient eclipses throw much more light on ancient chronology than might be inferred from the paper by Professor McFarland which was read before this Section at the Columbus meeting. The second paper is intended to show that there are secular equations of the mean distances of the planets from the sun. The third paper is in abstract as follows:

The Laplace coefficients of planetary perturbation arise from the development of the function

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* If the commutators of a group are self-conjugate the group is called *metabelian*. The group of cogredient isomorphisms of such a group is abelian, and conversely.

$$\{1 + a^2 - 2a\cos\beta\}^{-s}$$

in an infinite series depending on the cosines of β and its multiples. If we assume that

$$\{1 + a^2 - 2a\cos\beta\}^{-s} = \tfrac{1}{2}b_s^{(0)} + b_s^{(1)}\cos\beta + b_s^{(2)}\cos 2\beta + b_s^{(3)}\cos 3\beta + \cdots + b_s^{(i)}\cos i\beta, \quad (1)$$

the coefficients $b_s^{(i)}$ are the Laplace coefficients required.

If we put

$$\frac{2a}{1 + a^2} = a', \quad (2)$$

we shall have

$$\{1 + a^2 - 2a\cos\beta\}^{-s} = (1 + a^2)^{-s}\{1 - a'\cos\beta\}^{-s}; \quad (3)$$

and if we now assume that

$$\{1 - a'\cos\beta\}^{-s} = \tfrac{1}{2}\dot{b}_s^{(0)} + \dot{b}_s^{(1)}\cos\beta + \dot{b}_s^{(2)}\cos 2\beta + \dot{b}_s^{(3)}\cos 3\beta + \cdots + \dot{b}_s^{(i)}\cos i\beta, \quad (4)$$

we shall have

$$b_s^{(i)} = \dot{b}_s^{(i)} \cdot \{1 + a^2\}^{-s}. \quad (5)$$

If we now develop the first member of equation (4) by the binomial theorem, we shall have an infinite series of terms which may readily be computed by means of a table of logarithms of the coefficients of the different powers of $\cos\beta$, together with a similar table of logarithms of the coefficients $\cos^n\beta$ when it is developed in terms of the cosines of β and its multiples.

Tables containing the logarithms of the coefficients of these functions have been computed, so that we may easily compute all values of $\dot{b}_s^{(i)}$ for all values of $i = 0$, to $i = 35$. And when $\dot{b}_s^{(i)}$ has been found $b_s^{(i)}$ may be found by equation (5).

G. A. MILLER.

CORNELL UNIVERSITY.

NOTE ON GEOMETRY OF FOUR DIMENSIONS.

BY PROFESSOR E. O. LOVETT.

(Read before the American Mathematical Society, April 28, 1900.)

1. Speculations relative to the geometry of n dimensional space have followed several fairly well-defined trends which not infrequently cross one another: 1° A direct extension of the Cartesian geometry, which extension is to be regarded as nothing more than a convenient form of phraseology; in this form did n dimensional space spring forth from the minds of Grassmann, Cayley, Gauss, and Cauchy, and the idea was likely familiar to Euler and Lagrange. 2° The transformation of the ordinary visualizable spaces of two and three dimensions into manifoldnesses of higher or lower dimensions by introducing space elements other than the point or its dual element; for example, the line geometry of Plücker, the sphere geometry of Lie, the five dimensional manifoldness of all conics in the plane as an auxiliary to Ball's theory of screws; this category becomes more concrete perhaps than any other. 3° The absolute geometry of space; here would appear the celebrated dissertation of Riemann, the well-known memoirs of Helmholtz and Lie and the elaborate treatise of Veronese. 4° The extension of the methods of ordinary differential geometry to spaces of many dimensions; to this class belong the works of Christoffel, Beltrami, Bianchi, Cesàro, and Ricci, and the quite recent contributions of Darboux and his pupils. 5° The direct extension of the concepts and problems of metrical and projective geometry of ordinary space, as exemplified in the memoirs of Jordan, d'Ovidio, and Veronese. 6° The theory of birational correspondences between n dimensional aggregates as studied by Noether, Kantor, and Brill. 7° The descriptive geometry of space of n dimensions as begun in the papers of Veronese, Stringham, Schlegel, and Segre. 8° The kinematics of higher spaces as developed by Jordan, Clifford, and Beltrami. 9° The interpretation of n dimensional geometry in the light of the theory of groups as exhibited by Lie, Klein, and Poincaré.

2. It is proposed here to make an expository contribution to the ninth and fourth of the above categories, constructing ordinary four dimensional space by the method of Lie's theory of continuous groups, and studying curves of triple

curvature by the intrinsic analysis developed by Cesàro in his Lezioni di geometria intrinseca.

3. With regard to the space it is assumed:

1° That it is a four dimensional manifoldness, *i. e.*, that four independent data are necessary and sufficient to determine the position of an element of the manifoldness; these four independent things are called the coördinates of the element.

2° That a figure of the manifoldness possesses ten degrees of freedom within the manifoldness; *i. e.*, that ten independent data are necessary and sufficient to render a rigid body fixed in position; the latter ten independent things are called the parameters of the figure.

For convenience let the element be called a point, and its coördinates be designated by x, y, z, t. Consider any figure containing this point and let its parameters be $a_1, a_2, \cdots, a_{10}$. Let the figure assume a new position and call x', y', z', t' the coördinates of the new position of (x, y, z, t). Then

$$x' = \xi(x, y, z, t, a_1, a_2, \cdots, a_{10}), \quad y' = \eta(x, y, z, t, a_1, a_2, \cdots, a_{10}),$$

$$z' = \zeta(x, y, z, t, a_1, a_2, \ldots, a_{10}), \quad t' = \tau(x, y, z, t, a_1, a_2, \cdots, a_{10}).$$

The operation changing x, y, z, t into ξ, η, ζ, τ represents one of the motions of a four dimensional figure, and the ensemble of all these operations constitutes a continuous group with ten parameters. The identical transformation ought to appear among these operations. There must then be a system of the parameters $a_1, a_2, \cdots, a_{10}$ such that

$$\xi = x, \quad \eta = y, \quad \zeta = z, \quad \tau = t.$$

It is legitimate to assume the preceding system of values to be

$$a_1 = a_2 = \cdots = a_{10} = 0.$$

An infinitesimal transformation of the group is one whose parameters differ by infinitesimals from those giving the identical transformation; in this case the infinitesimal transformation is obtained by assigning infinitesimal values to the parameters themselves; *i. e.*, by such a transformation x, y, z, t are changed to

$$x + \sum_1^{10} a_i \frac{\partial \xi}{\partial a_i}, \quad y + \sum_1^{10} a_i \frac{\partial \eta}{\partial a_i}, \quad z + \sum_1^{10} a_i \frac{\partial \zeta}{\partial a_i}, \quad t + \sum_1^{10} a_i \frac{\partial \tau}{\partial a_i},$$

in the partial derivatives of which the a's should be put equal to zero.

Writing for short

$$p \equiv \frac{\partial f}{\partial x}, \quad q \equiv \frac{\partial f}{\partial y}, \quad r \equiv \frac{\partial f}{\partial z}, \quad s \equiv \frac{\partial f}{\partial t},$$

Lie's symbol for the above infinitesimal transformation is

$$I \equiv p \sum_1^{10} a_i \frac{\partial \xi}{\partial a_i} + q \sum_1^{10} a_i \frac{\partial \eta}{\partial a_i} + r \sum_1^{10} a_i \frac{\partial \zeta}{\partial a_i} + s \sum_1^{10} a_i \frac{\partial \tau}{\partial a_i}.$$

Putting

$$J_i \equiv p \frac{\partial \xi}{\partial a_i} + q \frac{\partial \eta}{\partial a_i} + r \frac{\partial \zeta}{\partial a_i} + s \frac{\partial \tau}{\partial a_i}, \quad (i = 1, 2, \cdots, 10)$$

any infinitesimal transformation is written

$$I \equiv \sum_1^{10} a_i J_i.$$

By a fundamental theorem of Lie, if we put

$$(J_i, J_j) \equiv \frac{\partial J_i}{\partial p}\frac{\partial J_j}{\partial x} - \frac{\partial J_i}{\partial x}\frac{\partial J_j}{\partial p} + \cdots + \frac{\partial J_i}{\partial s}\frac{\partial J_j}{\partial t} - \frac{\partial J_i}{\partial t}\frac{\partial J_j}{\partial s},$$

then

$$(J_i, J_j) = \sum_{k=1}^{k=10} \lambda_k^{(i,j)} J_k, \quad (i = 1, \cdots, 10\,;\ j = 1, \cdots, 10) \tag{1}$$

where the λ's are constants. There are forty-five of these equations, but the four hundred and fifty λ's are not wholly arbitrary, since the following one hundred and twenty identities of Jacobi must hold

$$(J_i, (J_j, J_k)) + (J_j, (J_k, J_i)) + (J_k, (J_i, J_j)) = 0, \quad (i, j, k = 1, 2, \cdots, 10). \tag{2}$$

Every set of J's satisfying (1) and (2) reveals a space whose independent infinitesimal motions are represented by the infinitesimal operations of the set. Those functions of the elements that are invariant under these transformations will characterize the geometry of the space.

4. It may be verified without difficulty that the following forms for the fundamental transformations $J_1, J_2, \cdots, J_{10}$, which are functions of x, y, z, t, p, q, r, s, namely

$$\begin{matrix} & xq - yp, & yr - zq, & xs - tp, \\ p, q, r, s, & & & \\ & zp - xr, & tr - zs, & ys - tq, \end{matrix} \tag{3}$$

satisfy the conditions (1) and (2).

Let (x, y, z, t) and (x', y', z', t') be any two points, and (x_1, y_1, z_1, t_1), (x_1', y_1', z_1', t_1') be their positions after they are subjected to the transformation

$$I \equiv \sum_1^{10} a_i J_i,$$

where the J_i's have the value (3).

Let $\varphi(x, y, z, t, x', y', z', t')$ be a function which is absolutely invariant under this operation; then must

$$\varphi(x_1, y_1, z_1, t_1, x_1', y_1', z_1', t_1') = \varphi(x, y, z, t, x', y', z', t'),$$

that is

$$I\varphi \equiv 0$$

for all values of the a_i's; hence equating to zero the coefficients of the a_i's in $I\varphi$ we have the following system of linear partial differential equations for the function φ:

$$\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial x'} = 0, \quad \frac{\partial\varphi}{\partial y} + \frac{\partial\varphi}{\partial y'} = 0, \quad \frac{\partial\varphi}{\partial z} + \frac{\partial\varphi}{\partial z'} = 0,$$

$$\frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial t'} = 0,$$

$$x\frac{\partial\varphi}{\partial y} - y\frac{\partial\varphi}{\partial x} + x'\frac{\partial\varphi}{\partial y'} - y'\frac{\partial\varphi}{\partial x'} = 0, \quad y\frac{\partial\varphi}{\partial z} - z\frac{\partial\varphi}{\partial y} + y'\frac{\partial\varphi}{\partial z'} - z'\frac{\partial\varphi}{\partial y'} = 0,$$

$$x\frac{\partial\varphi}{\partial t} - t\frac{\partial\varphi}{\partial x} + x'\frac{\partial\varphi}{\partial t'} - t'\frac{\partial\varphi}{\partial x'} = 0,$$

$$z\frac{\partial\varphi}{\partial x} - x\frac{\partial\varphi}{\partial z} + z'\frac{\partial\varphi}{\partial x'} - x'\frac{\partial\varphi}{\partial z'} = 0, \quad t\frac{\partial\varphi}{\partial z} - z\frac{\partial\varphi}{\partial t} + t'\frac{\partial\varphi}{\partial z'} - z'\frac{\partial\varphi}{\partial t'} = 0,$$

$$y\frac{\partial\varphi}{\partial t} - t\frac{\partial\varphi}{\partial y} + y'\frac{\partial\varphi}{\partial t'} - t'\frac{\partial\varphi}{\partial y'} = 0.$$

This system of ten equations in eight variables should not in general possess a solution. However the equations are not all independent. If the first seven be multiplied respectively by the following functions

$$0, \quad 0, \quad \frac{xy' - x'y}{y - y'}, \quad 0, \quad \frac{z't - zt'}{yt' - y't}, \quad \frac{x' - x}{y - y'}, \quad \frac{y'z - yz'}{yt' - y't},$$

and the results be added, the eighth equation of the system is obtained; similarly, if the multipliers be

$$0, \quad 0, \quad \frac{yt'-y't}{y-y'}, \quad \frac{x't-xt'}{x-x'}, \quad 0, \quad \frac{t-t'}{y-y'}, \quad \frac{z'-z}{x-x''}$$

the ninth equation results; and finally if the multipliers

$$0, \quad 0, \quad 0, \quad \frac{xy'-x'y}{x-x'}, \quad \frac{xt'-z't}{xz'-x'z'}, \quad \frac{xt'-x't}{xz'-x'z'}, \quad \frac{y-y'}{x-x''}$$

be taken, the summation yields the tenth equation of the system.

The system composed of the first seven equations possesses at least one solution; that it has no more is readily seen by noting the fact that not all seventh order determinants of the matrix

$$\begin{Vmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ y'-y & x-x' & 0 & 0 & -y' & x' & 0 & 0 \\ 0 & z'-z & y-y' & 0 & 0 & -z' & y' & 0 \\ t'-t & 0 & 0 & x-x' & -t' & 0 & 0 & x' \end{Vmatrix}$$

vanish; for example, the one formed by the last seven columns, whose value is written down at once as $(x-x')^2(y-y')$.

The unique solution of this system appears by observing that the first four equations demand that the solution be a function only of

$$X \equiv x-x', \quad Y \equiv y-y', \quad Z \equiv z-z', \quad T \equiv t-t';$$

the last three equations become

$$X\frac{\partial\varphi}{\partial Y} - Y\frac{\partial\varphi}{\partial X} = 0, \quad Y\frac{\partial\varphi}{\partial Z} - Z\frac{\partial\varphi}{\partial Y} = 0, \quad X\frac{\partial\varphi}{\partial T} - T\frac{\partial\varphi}{\partial X} = 0,$$

which require that φ be of the form

$$X^2 + Y^2 + Z^2 + T^2;$$

that is, the function

$$\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2 + (t-t')^2} \qquad (4)$$

is an absolute invariant under the most general transformaton of the group (3). This function is said to define the distance between the two points (x, y, z, t), (x', y', z', t').

5. Consider now the linear element manifoldness

$$\begin{aligned} x + lt + a &= 0, \\ y + mt + b &= 0, \\ z + nt + c &= 0. \end{aligned} \tag{5}$$

It is proposed to find the variations experienced by l, m, n, a, b, c, under the operations (3). This determination is readily effected in the following manner:

The total variations of (5) give

$$\delta(x + lt + a)=0, \quad \delta(y+mt+b)=0, \quad \delta(z+nt+c)=0. \tag{6}$$

In (6) are substitued successively the increments assigned to x, y, z, t by the transformations (3). A comparison of the resulting equations with the equations (5) gives equations for the variations of l, m, n, a, b, c. The results are tabulated below:

Transformation.	δl	δm	δn	δa	δb	δc
p	0	0	0	-1	0	0
q	0	0	0	0	-1	0
r	0	0	0	0	0	-1
s	0	0	0	$-l$	$-m$	$-n$
$xq-yp$	$-m$	l	0	$-b$	a	0
$yr-zg$	0	$-n$	m	0	$-c$	b
$xs-tp$	l^2+1	lm	ln	al	am	an
$zp-xr$	n	0	$-l$	c	0	$-a$
$tr-zs$	$-ln$	$-mn$	$-(1+n^2)$	$-cl$	$-cm$	$-cn$
$ys-tq$	lm	$1+m^2$	mn	bl	bm	bn

Two linear manifoldnesses (l, m, n, a, b, c), (l' m', n', a', b', c') possess invariants under the preceding transformations. These appear as solutions of the simultaneous system composed of the ten equations

$$\frac{\partial\varphi}{\partial l}\delta l + \frac{\partial\varphi}{\partial m}\delta m + \frac{\partial\varphi}{\partial n}\delta n + \frac{\partial\varphi}{\partial a}\delta a + \frac{\partial\varphi}{\partial b}\delta b + \frac{\partial\varphi}{\partial c}\delta c + \frac{\partial\varphi}{\partial l'}\delta l' + \cdots = 0,$$

formed by assigning successively to the variations the values given in the above table. This system has two independent solutions. These are analogous in form to the expression for the distance between two straight lines in ordinary space. There exists however another unique invariant of the two linear manifoldnesses, namely

$$m\frac{\partial\varphi}{\partial l}-l\frac{\partial\varphi}{\partial m}+m'\frac{\partial\varphi}{\partial n'}-l'\frac{\partial\varphi}{\partial m'}=0,$$

$$n\frac{\partial\varphi}{\partial m}-m\frac{\partial\varphi}{\partial n}+n'\frac{\partial\varphi}{\partial m'}-m'\frac{\partial\varphi}{\partial n'}=0,$$

$$l\frac{\partial\varphi}{\partial n}-n\frac{\partial\varphi}{\partial l}+l'\frac{\partial\varphi}{\partial n'}-n'\frac{\partial\varphi}{\partial l'}=0,$$

$$(l^2+1)\frac{\partial\varphi}{\partial l}+lm\frac{\partial\varphi}{\partial m}+ln\frac{\partial\varphi}{\partial n}+(l'^2+1)\frac{\partial\varphi}{\partial l'}$$
$$+l'm'\frac{\partial\varphi}{\partial m'}+l'n'\frac{\partial\varphi}{\partial n'}=0,$$

$$lm\frac{\partial\varphi}{\partial l}+(m^2+1)\frac{\partial\varphi}{\partial m}+nm\frac{\partial\varphi}{\partial n}+l'm'\frac{\partial\varphi}{\partial l'}$$
$$+(m'^2+1)\frac{\partial\varphi}{\partial m'}+m'n'\frac{\partial\varphi}{\partial n'}=0,$$

$$ln\frac{\partial\varphi}{\partial l}+mn\frac{\partial\varphi}{\partial m}+(n^2+1)\frac{\partial\varphi}{\partial n}+l'n'\frac{\partial\varphi}{\partial l'}+m'n\frac{\partial\varphi}{\partial m'}$$
$$+(n'^2+1)\frac{\partial\varphi}{\partial n'}=0,$$

which possesses a solution since the determinant of the coefficients is zero.

From the first three equations it is seen that φ is a function of

$$l^2+m^2+n^2,\quad l'^2+m'^2+n'^2,\quad ll'+mm'+nn';$$

calling these respectively u, v, and w, the last three equations become

$$2lu'\frac{\partial\varphi}{\partial u'}+2l'v'\frac{\partial\varphi}{\partial v'}+(l+l')w'\frac{\partial\varphi}{\partial w'}=0,$$

$$2mu'\frac{\partial\varphi}{\partial u'}+2m'v'\frac{\partial\varphi}{\partial v'}+(m+m')u'\frac{\partial\varphi}{\partial w'}=0,$$

$$2nu'\frac{\partial\varphi}{\partial u'}+2n'v'\frac{\partial\varphi}{\partial v'}+(n+n')w'\frac{\partial\varphi}{\partial w'}=0,$$

where

$$u'=u+1,\quad v'=v+1,\quad w'=w+1,$$

and, as should be the case, the determinant of the coefficients is zero. From any one of the three we have

$$\frac{du'}{u'} + \frac{dv'}{v'} = \frac{dw'}{w'};$$

that is, the solution φ is of the form $\frac{w'^2}{u'v'}$

or

$$\frac{(ll' + mm' + nn' + 1)^2}{(l^2 + m^2 + n^2 + 1)(l'^2 + m'^2 + n'^2 + 1)} = \cos^2\theta, \quad (7)$$

and the invariant θ is defined to be the angle between the two linear manifoldnesses.

The fundamental notions of distance and direction are thus introduced by the invariants (4) and (7), the former relative to two elements of the space, the latter relative to two simplest manifoldnesses composed of a simply infinite number of these elements. All the derived notions of geometry may then be developed by as simple extensions of these two primary notions from three dimensions to four dimensions as occur when passing from the plane to ordinary space.

6. A curve in a flat space of four dimensions is said to have triple curvature when not every five consecutive points of it lie in a euclidean space of three dimensions. Such a curve possesses not only rectilinear tangents, and osculating planes, but also osculating spaces of three dimensions; by the latter is meant the limiting position of a secant three-dimensional space Σ_3, $MM'M''M'''$ as the three points M', M'', M''' approach coincidence with M. At every point M of the curve we have to consider a tetrarectangular tetraeder which has as edges the tangent, the principal normal, the principal binormal, and the trinormal. By the principal normal is meant the only one of the ∞^1 normals at every point which lies in the osculating plane; by the principal binormal is meant the only one of the ∞^1 binormals at every point which lies in the osculating space; the trinormal is the only binormal perpendicular to the osculating space.

Taking the origin of the coördinate system at a point M movable along the curve, assume the tangent as x-axis, the trinormal as y-axis, the principal binormal as z-axis, and the principal normal as t-axis. Let $\delta\varphi$, $\delta\psi$, $\delta\chi$ be the respective angles between two consecutive tangents, two adjacent principal binormals, and two adjacent trinormals, when the origin is shifted from M to a neighboring point M' on the curve. The quadrangle of the direction cosines of the new axes referred to the old becomes

$$\begin{array}{c|cccc} & x & y & z & t \\ \hline x' & 1 & 0 & 0 & d\varphi \\ y' & 0 & 1 & d\chi & 0 \\ z' & 0 & -d\chi & 1 & d\psi \\ t' & -d\varphi & 0 & -d\psi & 1 \end{array} \qquad (11)$$

The limits of the ratios of $\delta\varphi$, $\delta\psi$, $\delta\chi$ to MM' as M' approaches M measure the curvatures of the curve at M; the first limit is called the flexion, the second the torsion, and the third the curvature; or perhaps more simply, the first, second, and third curvatures. Then if we put

$$ds = \rho d\varphi = \tau dx = r d\chi,$$

the numbers ρ, τ, r, the reciprocals of the curvatures, measure three lengths which are called the radii of the respective curvatures.

The array (11) shows that in order to discuss the curve in the domain of any of its points it is sufficient to know the functions φ, ψ, χ; that is, it is sufficient to give ρ, τ, r as functions of s. Three equations, the so-called intrinsic equations

$$f_1(\rho, \tau, r, s) = 0, \quad f_2(\rho, \tau, r, s) = 0, \quad f_3(\rho, \tau, r, s) = 0$$

which determine the radii of curvature as functions of s, determine at the same time the forms of those arcs of the curve which lie between points at which the tangent may become indeterminate; they do not fix the position of these branches in space.

7. The method of the intrinsic analysis employed so successfully by Professor E. Cesàro in his Lezioni di geometria intrinseca and developed by him for curves and surfaces in spaces of many dimensions may be exhibited for curves of triple curvature in detail in the following manner.

Let P be a point in space referred to M as origin and generally movable with M; let its coördinates be $x(s)$, $y(s)$ $z(s)$, $t(s)$. Let P' be the point of the trajectory of P corresponding to M'; its coördinates referred to the axes at M are $x + \delta x$, $y + \delta y$, $z + \delta z$, $t + \delta t$; its coördinates referred to the axes at M' are $x + dx$, $y + dy$, $z + dz$, $t + dt$; finally let u, v, w, p be the coördinates of M' referred to M. Then projecting $M'P'$ on the axes at M, we have by virtue of the array (11):

$$\begin{aligned} x + \delta x &= u + x + d\chi - (t + dt)\, d\varphi, \\ y + \delta y &= v + y + dy - (z + dz)\, d\chi, \\ z + \delta z &= w + z + dz + (y + dy)\, d\chi - (t + dt)\, d\psi, \\ t + \delta t &= p + t + dt + (x + da)\, d\varphi + (z + dz)\, d\psi. \end{aligned} \tag{12}$$

Excluding from further consideration here those curves for which it is not legitimate to assume that the limit of the ratio of the arc to the arc is unity, we have from the definition of the tangent as the limiting position of a secant

$$\pounds \frac{u}{\delta s} = 1, \qquad \pounds \frac{v}{\delta s} = \pounds \frac{w}{\delta s} = \pounds \frac{p}{\delta s} = 0,$$

and the formulæ (12) give Cesàro's fundamental relations

$$\begin{aligned} \frac{\delta x}{ds} &= \frac{dx}{ds} - \frac{t}{\rho} + 1, \quad \frac{\delta y}{ds} = \frac{dy}{ds} - \frac{z}{r}, \\ \frac{\delta z}{ds} &= \frac{dz}{ds} + \frac{y}{r} - \frac{t}{\tau}, \quad \frac{\delta t}{ds} = \frac{dt}{ds} + \frac{x}{\rho} + \frac{z}{\tau}. \end{aligned} \tag{13}$$

The formulæ (12) are also applicable to the cosines α, β, γ, ε that define any direction, provided that u, v, w, p are zero, and the corresponding fundamental formulæ for this direction are found by replacing x, y, z, t in (13) respectively by α, β, γ, ε.

From (13) the necessary and sufficient conditions for the immobility of a point are

$$\frac{dx}{ds} = \frac{t}{\rho} - 1, \quad \frac{dy}{ds} = \frac{z}{r}, \quad \frac{dz}{ds} = \frac{t}{\tau} - \frac{y}{r}, \quad \frac{dt}{ds} = -\frac{x}{\delta} - \frac{\gamma}{\tau}. \tag{14}$$

8. The above fundamental formulæ may be employed to show that every trio of intrinsic equations determines a curve uniquely, at least between convenient limits of s in a range where the curvatures are finite and continuous functions of the arc. In fact there exists, by the theory of systems of linear differential equations, one and but one quatrain of functions $x(s)$, $y(s)$, $z(s)$, $t(s)$ satisfying the simultaneous system (14), and reducing to a, b, c, e for $s = 0$. These functions x, y, z, t are evidently the coördinates with regard to the tetraeder of origin M of that point whose coördinates with regard to the tetraeder of origin O are a, b, c, e.

As remarked above, the necessary and sufficient con-

ditions for the invariability of the direction $(\alpha, \beta, \gamma, \epsilon)$ are

$$\frac{dx}{ds}=\frac{\epsilon}{\rho}, \quad \frac{d\beta}{ds}=\frac{\gamma}{r}, \quad \frac{d\gamma}{ds}=-\frac{\beta}{r}+\frac{\epsilon}{\tau}=, \quad \frac{d\epsilon}{dt}=-\frac{\alpha}{\rho}-\frac{\gamma}{\tau}; \qquad (15)$$

if $(\alpha, \beta, \gamma, \epsilon,)$ and $(\alpha', \beta', \gamma', \epsilon')$ are any two solutions of this system it follows from the equations of the system that

$$\frac{d}{ds}(\alpha\alpha' + \beta\beta' + \gamma\gamma' + \epsilon\epsilon') = 0, \quad \frac{d}{ds}(\alpha^2 + \beta^2 + \gamma^2 + \epsilon^2) = 0,$$

$$\frac{d}{ds}(\alpha'^2 + \beta'^2 + \gamma'^2 + \epsilon'^2) = 0;$$

that is

$$S\alpha\alpha', \qquad S\alpha^2, \qquad S\alpha'^2$$

are integrals of (15).

Hence if there be determined four systems of functions contained in the first of the following arrays

α_1	β_1	γ_1	ϵ_1	1	0	0	0
α_2	β_2	γ_2	ϵ_2	0	1	0	0
α_3	β_3	γ_3	ϵ_3	0	0	1	0
α_4	β_4	γ_4	ϵ_4	0	0	0	1

in such a manner that they satisfy (15) and take the corresponding values in the second array for $s = 0$, we have

$$S\alpha_i\alpha_j = 0, \quad (i \neq j); \quad S\alpha_i\alpha_j = 1, \quad (i = j);$$

that is, the determinant represented by the first array is orthogonal. The elements of this determinant are the direction cosines of axes of origin O with respect to those of origin M, since they give the directions of the former for $s = 0$, and satisfy the conditions (15) for the invariability of those directions. Then in all cases

$$x = x_0 + a\alpha_1 + b\alpha_2 + c\alpha_3 + e\alpha_4, \quad y = y_0 + a\beta_1 + b\beta_2 + c\beta_3 + e\beta_4,$$

$$z = z_0 + a\gamma_1 + b\gamma_2 + c\gamma_3 + e\gamma_4, \quad t = t_0 + a\epsilon_1 + b\epsilon_2 + c\epsilon_3 + e\epsilon_4;$$

for each value of s the constants a, b, c, e, can be determined in such manner that they represent the coördinates of M with respect to axes of origin O; for this purpose it is sufficient to put x, y, z, t equal to zero in the preceding equations and solve for a, b, c, e:

$$a=-(\alpha_1x_0+\beta_1y_0+\gamma_1z_0+\epsilon_1t_0), \quad b=-(\alpha_2x_0+\beta_2y_0+\gamma_2z_0+\epsilon_2t_0),$$
$$c=-(\alpha_3x_0+\beta_3y_0+\gamma_3z_0+\epsilon_3t_0), \quad e=-(\alpha_4x_0+\beta_4y_0+\gamma_4z_0+\epsilon_4t_0). \qquad (16)$$

Thus the curve defined by a trio of intrinsic equations is uniquely determined; by varying s the coördinates (a, b, c, e) of all of its points with regard to a system of immovable axes are known.

9. It may be observed in conclusion that the whole theory of curves of triple curvature may be deduced by differentiation and elimination by means of the fundamental invariance conditions (14) and (15).

Differentiating (16) we have

$$\frac{da}{ds} = -S\left(x_0 \frac{dx_0}{ds} + a_1 \frac{dx_0}{ds}\right) = a_1, \quad \frac{db}{ds} = a_2, \quad \frac{dc}{ds} = a_3, \quad \frac{de}{ds} = a_4;$$

also

$$\frac{d^2a}{ds^2} = \frac{\varepsilon_1}{\rho}, \cdots;$$

$$\frac{d^3a}{ds^3} = \varepsilon_1 \frac{d}{ds}\frac{1}{\rho} - \frac{a_1}{\rho^2} - \frac{\gamma_1}{\rho\tau}, \cdots.$$

These formulæ lead to the following expressions for the curvatures as functions of the arc s: for the first curvature, or flexion, which measures the rapidity with which the curve deviates from being a straight line,

$$\frac{1}{\rho^2} = \left(\frac{d^2a}{ds^2}\right)^2 + \left(\frac{d^2b}{ds^2}\right)^2 + \left(\frac{d^2c}{ds^2}\right)^2 + \left(\frac{d^2e}{ds^2}\right)^2;$$

for the second curvature, or torsion, which expresses the rapidity with which the curve deviates from being a plane curve,

$$\frac{1}{\tau^2} = \frac{S\left(\frac{d^2a}{ds^2}\right)^2 S\left(\frac{d^3a}{ds^3}\right)^2 - \left\{S\left(\frac{d^2a}{ds^2}\right)^2\right\}^3 - \left\{S\frac{d^2a}{ds^2}\frac{d^3a}{ds^3}\right\}^2}{\left\{S\left(\frac{d^2a}{ds^2}\right)^2\right\}^2};$$

finally for the third curvature, which measures the rapidity of deviation from a curve of double curvature,

$$\frac{1}{r^2} = \left(\frac{dA}{ds}\right)^2 + \left(\frac{dB}{ds}\right)^2 + \left(\frac{dC}{ds}\right)^2 + \left(\frac{dE}{ds}\right)^2,$$

where $A = A'/D$, $B = B'/D$, $C = C'/D$, $E = E'/D$,

$$D^2 = A'^2 + B'^2 + C'^2 + E'^2,$$

A', B', C', E' being the minors corresponding to a, b, c, e in the Wronskian

$$W = |a, \quad \frac{db}{ds}, \quad \frac{d^2c}{ds^2}, \quad \frac{d^3e}{ds^3}|.$$

The above values for the curvatures are derived directly by Pirondini in *Battaglini's Journal* for 1890.

PRINCETON, NEW JERSEY.

NOTES.

THE closing (October) number of Volume I of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "On surfaces enveloped by spheres belonging to a linear spherical complex," by P. F. SMITH; "On certain relations among the theta constants," by J. I. HUTCHINSON; "On groups which have the same group of isomorphisms," by G. A. MILLER; "Die Hesse'sche und die Cayley'sche Curve," by P. GORDAN; "Application of a method of D'Alembert to the proof of Sturm's theorem of comparison," by M. BÔCHER; "Two plane movements generating quartic scrolls," by E. M. BLAKE; "The invariant theory of the inversion group: geometry upon a quadric surface," by E. KASNER; "A simple proof of the fundamental Cauchy-Goursat theorem." by E. H. MOORE; "Notes and errata: Volume I."

THE mathematical section of the British association for the advancement of science held its meeting at Bradford, September 10, 1900, with Major P. A. MacMahon, F.R.S. as presiding officer, and Mr. E. T. Whittaker, secretary. The following papers were read:

(1) Miss F. HARDCASTLE: Report (preliminary) on the present state of the theory of point groups.

(2) MAJOR P. A. MACMAHON: "A property of the characteristic symbolic determinant of any n quantics in n variables."

(3) PROFESSOR CYPARISSOS STEPHANOS: "Sur les relations entre la géométrie projective et la mécanique."

(4) Mr. H. S. CARSLAW: "The use of multiple space in applied mathematics."

(5) MAJOR P. A. MACMAHON: "The asyzygetic and perpetuant covariants of systems of binary quantics."

(6) LIEUT.-COL. ALLAN CUNNINGHAM: "Determination of successive high primes."

(7) Mr. A. B. BASSET: "A quintic curve cannot have more than fifteen real points of inflexion."

(8) MAJOR P. A. MACMAHON: "On the symbolism appropriate to the study of orthogonal and Boolian invariant systems which appertain to binary and other quantics."

(9) DR. J. WILLIS: "On the construction of magic squares."

(10) PROFESSOR J. D. EVERETT: "On a central-difference interpolation formula."

(11) PROFESSOR EVERETT: "On Newton's contributions to central-difference interpolation."

The committee appointed in 1888 to calculate tables of certain mathematical functions presented a report of their year's progress. The work on which they have for some time been engaged, namely, the preparation of a new "Canon Arithmeticus," is now almost completed. The calculations were made by Lieut.-Col. ALLAN CUNNINGHAM.

FOR volumes IV. and V. of the Encyclopædia of the mathematical sciences, edited by H. BURKHARDT, and W. FR. MEYER (see BULLETIN, 2d series, volume 3, p. 326; volume 4, p. 32; volume 5, pp. 109, 151, 202, 266); the following list of collaborators and subjects is announced:

Volume IV: Mechanics.—*A.* Introduction: 1. The principles of rational mechanics, A. Voss, of Würzburg.—*B.* Mechanics of Points and Rigid Systems: I. Geometric foundation and development of concrete modes of representation: 2. Theory of systems of vectors and screws, H. E. TIMERDING, of Strassburg; 3, Geometry of masses, G. JUNG, of Milan; 4. Kinematics. A. SCHOENFLIESS, of Königsberg; 5. Elementary statics, including graphical statics, L. HENNEBERG, of Darmstadt; 6. Elementary kinetics, J. PETERSEN, of Copenhagen. II. General analytical treatment of arbitrary systems with finite degrees of freedom: 7. General developments, P. STÄCKEL, of Kiel; 8. Complete treatment of special cases, P. STÄCKEL of Kiel; 9. Mathematical treatment of the problem of n bodies, E. T. WHITTAKER, of Cambridge, Eng. III. Applications, with consideration of disturbing influences: 10. Mechanics of the simplest kinds of physical apparatus and of experiments, PH. FURTWÄNGLER, of Potsdam; 11. Kinetic problems of machines, K. HEUN, of Berlin; 12. Ballistics, C. CRANZ, of Stuttgart; 13. Sports, G. T. WALKER, of Cambridge, Eng.—*C.* Mechanics of Deformable Bodies. I. Analytical geometrical

aids: 14. Vector analysis, M. ABRAHAM, of Göttingen. II. Hydrodynamics: 15. Physical foundation, A. E. H. LOVE, of Oxford; 16. Theoretical developments, A. E. H. LOVE, of Oxford; 17. Hydraulics, part one: Flow of water in pipes, canals, etc., E. PALADINI, of Milan; 18. Hydraulics, part two: Motors and pumps, M. GRÜBLER, of Charlottenburg; 19. Motion of ships, A. KRILOFF, of St. Petersburg; 20. Aerodynamics, S. FINSTERWALDER, of Munich. III. Theory of elasticity: 21. Physical foundation, A. SOMMERFELD, of Aachen; 22. Theoretical treatment of statical problems, O. TEDONE, of Genoa; 23. Statics of building construction, E. OVAZZA, of Palermo; 24. Vibrations, in particular acoustics, H. LAMB, of Manchester, Eng.; 25. Theory of measuring apparatus based upon elastic effects, PH. FURTWÄNGLER, of Potsdam.—*D.* Mechanics of Systems Consisting of very Numerous Discrete Particles: 26. The application of the theory of probabilities, L. BOLTZMANN, of Leipsic.

Volume V: Physics.—*A.* Introduction: 1. Measures and measurement, C. RUNGE, of Hanover; 2. General properties of bodies, Gravity, J. ZENNECK, of Strassburg.—*B.* Thermodynamics: 3. General foundations of thermodynamics, G. H. BRYAN, of Upper Bangor, Wales; 4. Special substances and conditions, H. KAMERLINGH-ONNES, of Leiden, and J. KORTEWEG, of Amsterdam; 5. Dissipation of energy, including conduction of heat, E. W. HOBSON, of Cambridge, Eng.; 6. Technical theory of heat, H. LORENZ, of Halle, and E. RIECKE, of Göttingen.—*C.* Molecular Physics: 7. Fundamental notions concerning atoms and molecules; *a.* Introductory remarks on the atomic theory, L. BOLTZMANN, of Leipsic; *b.* The mathematical foundations of chemistry, W. MEYERHOFFER, of Berlin; 8. Crystallography; *a.* Theory of symmetry and structure, A. SCHOENFLIESS and O. MÜGGE, of Königsberg; *b.* Computation and drawing of crystals, TH. LIEBISCH, of Göttingen; 9. Kinetic theory of gases, L. BOLTZMANN, of Leipsic; 10. Capillarity and cohesion, H. MINKOWSKI, of Zurich; 11. Physical chemistry and electrochemistry, J. H. VAN'T HOFF, of Berlin.—*D.* Electricity and Optics: Physical foundations of the theory of electricity. 12. Action at a distance, laws governing elements, R. REIFF, of Stuttgart; 13. Action of a field, Maxwell's theory and allied topics, H. A. LORENTZ, of Leiden; 14. Further developments of Maxwell's theory, Electron-theory, H. A. LORENTZ, of Leiden.—Special mathematical developments in the theory of electricity. 15. Electrostatics and magnetostatics, H. M.

MACDONALD, of Cambridge, Eng.; 16. Relations between electricity and elastic deformation, F. POCKELS, of Heidelberg; 17. Stationary and slowly changing fields (electric currents, induction, and electrodynamics in the narrower sense), A. TAUBER, of Vienna; 18. Relations of the electric current to heat and magnetism, H. DIESSELHORST, of Berlin; 19. Rapidly changing fields, M. ABRAHAM, of Göttingen; 20. Technical electricity, TH. DESCOUDRES, of Göttingen.—Physical foundations of optics. 21. Older theories, A. WANGERIN, of Halle; 22. Electromagnetic theory of light, W. WIEN, of Würzburg; 23. The part played by molecular physics and the electron-theory in optics, W. WIEN, of Würzburg.—Special mathematical developments in optics. 24. Ray treatment of optics and optical instruments, S. FINSTERWALDER, of Munich; 25. Wave treatment of optics (interference and refraction), K. STREHL, of Erlangen; 26. Optics of crystals, F. POCKELS, of Heidelberg.—*E.* Conclusion: 27. General physical notions and methods, A. SOMMERFELD, of Aachen, and J. LARMOR, of Cambridge, Eng.

CAMBRIDGE UNIVERSITY.—Advanced mathematical courses for the current academic year are announced as follows:

Michaelmas term, 1900:—By Professor Sir G. G. STOKES: Hydrodynamics, three hours.—By Professor A. R. FORSYTH: Abel's theorem and abelian functions, three hours; Invariants and covariants, two hours.—By Professor G. H. DARWIN: Lunar theory, three hours.—By Mr. R. PENDLEBURY: Theory of equations, three hours.—By Dr. E. W. HOBSON: Spherical and cylindrical harmonics, three hours.—By Mr. J. LARMOR: Electricity and magnetism, three hours.—By Mr. H. F. BAKER: Continuous groups, three hours.—By. Mr. H. M. MACDONALD: Waves (especially waves of light), three hours.—By Mr. H. W. RICHMOND: Plane analytical geometry, three hours.—By Mr. G. T. WALKER: The electro-magnetic field, three hours.—By Mr. E. T. WHITTAKER: The problem of three bodies, three hours.—By Mr. J. H. GRACE: Invariants and geometrical applications, three hours.

Lent term, 1901:—By Professor Sir G. G. STOKES: Physical optics, three hours.—By Professor A. R. FORSYTH: Invariants and covariants (symbolical methods), three hours.—Fourier's and other expansion theorems, two hours.—By Professor Sir R. S. BALL: Planetary theory, three hours.—By Mr. R. PENDLEBURY: Theory of numbers, three hours.—By Dr. E. W. HOBSON: Vibrations and

sound, three hours.—By Mr. J. LARMOR: Electricity and magnetism (continued, more advanced), three hours.—By Mr. H. F. BAKER: Discontinuous groups, three hours.—By Mr. H. M. MACDONALD: Elasticity, three hours.—By Dr. J. W. L. GLAISHER: Elliptic functions, three hours.—By Mr. A. BERRY: Elliptic functions, three hours.—By Mr. G. T. WALKER: Physical optics, three hours.—By Mr. G. T. BENNETT: Linear and quadratic complexes, three hours.—By Mr. E. T. WHITTAKER: Linear differential equations of the second order.

Easter term, 1901:—By Professor G. H. DARWIN: Figure of the earth and precession, three hours.—By Mr. W. L. MOLLISON: Theory of potential and electrostatics.—By Mr. A. N. WHITEHEAD: Non-euclidean geometry.—By Mr. G. T. WALKER: Rigid dynamics, three hours.—By Mr. I'A. BROMWICH: Algebra of matrices and bilinear forms.

Long vacation, 1901:—By Professor Sir R. S. BALL: Theory of screws, two hours.

THE several universities below offer during the winter semester 1900–1901 courses in mathematics as follows:

UNIVERSITY OF STRASSBURG.—By Professor T. REYE: Geometry of position, three hours; Analytical mechanics, two hours; Seminar, two hours.—By Professor H. WEBER: Elliptic functions, four hours; Selected chapters of higher algebra, two hours; Seminar in coöperation with Dr. WELLSTEIN.—By Professor G. ROTH: Algebraic analysis and determinants, three hours; Analytical geometry of space, two hours; Ordinary differential equations, two hours.—By Professor A. KRAZER: Infinitesimal calculus, four hours; Plane analytical geometry, three hours; Seminar, two hours.—By Dr. H. E. TIMERDING: Introduction to higher analysis, two hours; Elasticity and rigidity, one hour.—By Dr. J. WELLSTEIN: Descriptive geometry I. two hours, with exercises; Seminar, with Professor WEBER.

UNIVERSITY OF TÜBINGEN.—By Professor A. v. BRILL: Introduction to higher mathematics, four hours; On non-rigid systems and the mechanics of Hertz, three hours; Seminar, two hours.—By Professor H. STAHL: Higher algebra, three hours; application of the theory of functions, four hours; Seminar, two hours.—By Professor L. MAURER: Higher analysis II, two hours, with exercises, one hour; Spherical trigonometry, one hour, with exercises, one hour; Theory of numbers, two hours.

UNIVERSITY OF WÜRZBURG.—By Professor F. PRYM:

Differential calculus, four hours, with exercises, two hours; Higher theory of functions, four hours; Seminar in higher mathematics, two hours.—By Professor A. Voss: Introduction to the theory of differential equations, four hours; Application of the infinitesimal calculus to the theory of curves and surfaces, four hours; Seminar, in analytical and synthetical geometry of conics, two hours.

UNIVERSITY OF VIENNA.—By Professor J. v. ESCHERICH: Definite integrals and differential equations, five hours; Proseminar, one hour; Seminar, two hours.—By Professor L. GEGENBAUER: Algebra III, two hours; Theory of numbers, three hours; Proseminar, one hour; Seminar, two hours.—By Professor F. MERTENS: Elements of infinitesimal calculus for students of all faculties, with exercises, five hours; Proseminar, one hour; Seminar, two hours.—By Professor G. KOHN: Introduction to synthetic geometry, four hours, with exercises, one hour; Algebraic plane curves, one hour.—By Dr. V. SERSAWY: Mathematics of insurance, two courses of three and four hours, respectively.—By Dr. A. TAUBER: Analytical mechanics, three hours; Mathematics of insurance, four hours, with exercises, two hours.—By Dr. E. BLASCHKE: Introduction to mathematical statistics, three hours.—By Dr. K. ZSIGMONDY: Surfaces of second order, one hour.—By Dr. R. D. v. STERNECK: Differential geometry, three hours.

UNIVERSITY OF ZÜRICH.—By Professor H. BURKHARDT: Infinitesimal calculus, four hours; Differential equations, four hours; Seminar, two hours.—By Professor A. WEILER: Analytical geometry, with exercises, four hours; Descriptive geometry, with exercises, four hours; Synthetic geometry, two hours; Mathematical geography, two hours.—By Dr. F. KRAFT: Modern synthetic geometry, four hours; Analytical geometry, two hours; Analytical mechanics, four hours; Exercises in higher mathematics, two hours.—By Dr. E. GUBLER: Theory and resolution of higher equations, with exercises, three hours; Methods of mathematical instruction in secondary schools, two hours; Spherical trigonometry, one hour.

THE Göttingen Academy of Sciences has awarded 800 marks to Professor F. KLEIN for his work on the Mathematical Encyclopædia.

PROFESSOR W. DYCK has been appointed director of the Technical High School in Munich for the next three years.

DR. J. B. FAUGHT, recently assistant professor of mathe-

matics in the University of Indiana, has been appointed professor of mathematics in the Northern Michigan State Normal School, Marquette, Mich.

DR. D. F. CAMPBELL, of the Lawrence Scientific School, Harvard University, has been appointed instructor in mathematics in the Armour Institute of Technology, Chicago, Ill.

DR. G. A. BLISS has been appointed instructor in mathematics at the University of Minnesota.

MR. W. B. FORD has been appointed instructor in mathematics in the University of Michigan.

PROFESSOR F. H. LOUD has resumed his duties at Colorado College after two years' leave of absence.

PROFESSOR E. J. TOWNSEND has returned to his position at the University of Illinois after two years of study at the University of Göttingen.

PROFESSOR I. STRINGHAM has returned to the University of California, after a year of travel in Europe.

PROFESSOR B. O. PEIRCE, of Harvard University, has been granted a year's leave of absence.

PROFESSOR S. J. BROWN, astronomical director of the Naval Observatory, has been appointed superintendent of the Nautical Almanac, to succeed Professor H. T. TODD, who was retired by age on August 25th.

PROFESSOR R. E. ALLARDICE, of Leland Stanford University, is spending a year abroad on leave of absence.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ABEL (N. H.). Abhandlung über eine besondere Klasse algebraisch auflösbarer Gleichungen. (1829.) Herausgegeben von A. Loewy. Leipzig, Engelmann, 1900. 12mo. 50 pp. (Ostwald's Klassiker der exakten Wissenschaften, No. 111.) Cloth. M. 0.90

BEER (F.). Kriterien für die Irrazionalität von Funktionalwerten. (Diss.) Göttingen, 1899. 8vo. 61 pp.

BEINHORN (J.). Zur Theorie der quadratischen Formen. (Diss.) Marburg, 1900. 8vo. 42 pp.

BINDER (J. K.). Ueber eine gewisse Abbildung zweier Rotationshyperboloide auf einander. (Diss.) Leipzig, 1900. 8vo. 81 pp., 4 plates.

BOEHM (K.). Zur Integration partieller Differentialsysteme. (Habilitationsschrift.) Leipzig, Teubner, 1900. 8vo. 55 pp. M. 1.80

BORNEMANN (G.). Analytische Studien. (Progr.) Chemnitz, 1899. 4to. 28 pp.

BURONI (S.). See LAVAGGI.

CAUCHY (A. L.). Abhandlung über bestimmte Integrale zwischen imaginären Grenzen. (1825.) Herausgegeben von P. Stäckel. Leipzig, Engelmann, 1900. 12mo. 80 pp. (Ostwald's Klassiker der exakten Wissenschaften, No. 112.) Cloth. M. 1.25

———. See LAGRANGE.

ENGEL (F.). Sophus Lie. Ausführliches Verzeichnis seiner Schriften. Mit dem Bildnis Sophus Lies in Heliogravüre. Leipzig, Teubner, 1900. 8vo. 42 pp. M. 2.00

FABBRI (E.). Il teorema dell'integrale di Cauchy; contributo alla storia critica dell'analisi. Bologna, Zamorani e Albertazzi, 1900. 8vo. 71 pp.

FLECHSENHAAR (A.). Ueber Multiplizität von Gleichungen. (Diss.) Giessen, 1899. 8vo. 27 pp.

GUTZMER (A.), HAUCK (G.), HENSEL (K.), HEUN (K.). See JAHRESBERICHT.

JAHRESBERICHT der Deutschen Mathematiker-Vereinigung. Vol. 8, Heft 2, enthaltend: Die Entwickelung der Lehre von den Punktmannigfaltigkeiten. Bericht, erstattet der Deutschen Mathematiker-Vereinigung von A. Schoenflies. Herausgegeben im Auftrage des Vorstandes von G. Hauck und A. Gutzmer. Leipzig, Teubner, 1900. 8vo. 4 + 6 + 251 pp.

———. Vol. 9, Heft 2, enthaltend: Die kinetischen Probleme der wissenschaftlichen Technik. Bericht, erstattet der Deutschen Mathematiker-Vereinigung von K. Heun. Herausgegeben im Auftrage des Vorstandes von K. Hensel und A. Gutzmer. Leipzig, Teubner, 1900. 8vo. 6 + 123 pp.

KLEIBER (J.). Ueber die Priminvarianten quadratischer Formen beliebig hoher Stufe. (Progr.) München, 1900. 8vo. 58 pp.

KOWALEWSKI (G.). See LAGRANGE.

LAGRANGE und CAUCHY. Zwei Abhandlungen zur Theorie der partiellen Differenzialgleichungen erster Ordnung. (1772 und 1819.) Herausgegeben von G. Kowalewski. Leipzig, Engelmann, 1900. 12mo. 54 pp. (Ostwald's Klassiker der exakten Wissenschaften, No. 113.) Cloth. M. 1.00

LAVAGGI. Calcolo infinitesimale; lezioni [dettate nell'anno] 1899–1900 nella r. università di Parma, compilate per cura di S. Buroni. Parma, Zafferri, 1900. 8vo. Pp. 433–560.

LEITZMANN (H.). See PASCAL (E.).

LÉVY (L.). See ROUCHÉ (E.).

LOEWY (A.). See ABEL (N. H.).

MEINEL (K.). Ueber Potenzlinien und Potenzkreise, das Apollonische Taktionsproblem und die Malfattische Aufgabe. (Progr.) Fürth, 1900. 8vo. 33 pp., 3 plates.

PASCAL (E.). Repertorium der höheren Mathematik (Definitionen, Formeln, Theoreme, Literatur). Autorisirte deutsche Ausgabe nach einer neuen Bearbeitung des Originals von A. Schepp. In 2 Teilen: Analysis und Geometrie. Teil I: Analysis. Leipzig, Teubner, 1900. 8vo. 12 + 638 pp. Cloth. M. 10.00

———. Die Determinanten. Eine Darstellung ihrer Theorie und Anwendungen mit Rücksicht auf die Gesamtheit der neueren Forschungen. Deutsch von H. Leitzmann. Leipzig, Teubner, 1900. 8vo. 16 + 266 pp. M. 10.00

ROUCHÉ (E) et LÉVY (L.). Analyse infinitésimale, à l'usage des ingénieurs. Vol. 1: Calcul différentiel. Paris, Gauthier-Villars, 1900. 8vo. 8 + 559 pp. (Encyclopédie industrielle.)

SCHEPP (A.). See PASCAL (E.).

SCHERRER (O.) Ueber Kegelschnitte im Raume. (Progr.) Frauenfeld, 1900. 4to. 35 pp.

SCHILLING (F.). Ueber die Nomographie von M. d'Ocagne. Eine Einführung in dieses Gebiet. Leipzig, Teubner, 1900. 8vo. 47 pp. M. 2.00

SCHMIDT (R.). Beiträge zum Gesetze der kleinen Zahlen. (Diss.) Göttingen, 1900. 8vo. 56 pp.

SCHOENFLIES (A.). See JAHRESBERICHT.

STÄCKEL (P.). See CAUCHY (A. L.).

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